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In-plane/out-of-plane concentrated forces and moments on composite laminates with elliptical elastic inclusions

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Abstract

The problems of composite laminates containing elliptical elastic inclusions subjected to concentrated forces and moments are considered in this paper. By employing Stroh-like formalism for the coupled stretching–bending analysis, analytical closed form solutions are obtained explicitly. The generality of the solutions provided in this paper can be shown as follows: (1) The laminates include any kinds of laminate lay-ups, symmetric or unsymmetric, which allow the stretching and bending deformations couple each other. (2) The concentrated forces and moments can be applied in in-plane and/or out-of-plane directions, located inside and/or outside the inclusions. (3) The elliptical elastic inclusions can be any kinds of elastic materials including the limiting cases such as holes, rigid inclusions, cracks, line inclusions, etc. Since no such general solution has been found in the literature, the solutions are checked and verified by the special cases that no inclusions are embedded in the laminates, and that the inclusions are replaced by holes. Moreover, with various hardness ratios of inclusion and matrix some numerical examples showing the stress resultants along the interface are presented. Like the Green's functions for the infinite laminates and those containing holes/cracks, the present solutions associated with the in-plane concentrated forces and out-of-plane concentrated moments have exactly the same mathematical form as those of the corresponding two-dimensional problems, in which the only difference is the contents of the symbols. While for the other loading cases, new types of solutions are obtained explicitly.

Keywords: Green's function; Elliptical elastic inclusions; Composite laminates; Stretching–bending coupling; Stroh-like formalism; Concentrated forces and moments

1. Introduction

On account of the linear character of the related equations, the principle of superposition is applicable to most of the fundamental problems of elasticity. Thus, the solutions associated with the concentrated forces and moments, generally called *Green's functions*, become important in constructing general solutions through superposition. Because of its importance, many analytical solutions of Green's functions have been published

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in journals and books. Among them, some are related to the present topic – laminates with elliptical elastic inclusions. For example, Hwu and Yen (1993) obtained the Green's functions for elastic inclusions whose point loads are applied outside the inclusions. To consider the other possible locations of the point loads or dislocations, Yen et al. (1995) provided the solutions whose loads may be applied inside, outside or on the interface of the inclusions. Despite the generality of the solutions in terms of inclusions and locations, they are restricted to two-dimensional deformation and no bending deformation has been considered. Therefore, these solutions are not applicable to the cases in which the plate is subjected to bending moments or bending deformation occurs even under inplane loadings.

To consider the most general stretching–bending coupling problems of two-dimensional anisotropic plates, several different complex variable formulations have been proposed in the literature such as (Becker, 1991; Lu and Mahrenholtz, 1994; Cheng and Reddy, 2002; Yin, 2003a,b; Hwu, 2003a). Most of the formulations considered the nondegenerate anisotropic plates whose material eigenvalues are distinct. The fully discussion covering all types of anisotropic plates – degenerate or nondegenerate, with or without stretching–bending coupling was considered by Yin (2003a,b). Recently Yin (2005a,b) further provided the Green's functions of infinite plates, semi-infinite plates and the plates with elliptical holes or rigid inclusions for all types of degenerate or nondegenerate laminates. For nondegenerate laminates, some Green's functions related to holes and cracks are obtained by Hwu (2005). Although the elliptical inhomogeneity has been considered by (Cheng and Reddy, 2004), the applied loading is remote uniform membrane stress resultants and bending moments not the concentrated forces and moments. As far as we know no Green's functions for the laminates with *elastic* inclusions have been found in the literature neither nondegnerate nor degenerate when both the stretching and bending deformations are allowed to occur in the laminates.

To find the Green's functions of the laminates with elliptical elastic inclusions, in this paper Stroh-like formalism for the stretching–bending coupling analysis (Hwu, 2003a) was employed. This formalism has been successfully applied to the cases of the infinite laminates (Hwu, 2004) and the laminates with holes/cracks subjected to inplane/out-of-plane concentrated forces and moments (Hwu, 2005). From these studies we observe that without considering the out-of-plane concentrated forces and inplane torsion the solutions will keep the same mathematical forms as their corresponding two-dimensional problems, which was also indicated by Yin (2005a,b). However, the key loading that distinguishes the inplane problem from the plate bending problem is the out-of-plane force, and hence it cannot be missed in the coupled stretching–bending analysis. To have a complete solution for the present problem, all inplane and out-of-plane concentrated forces and moments will be considered. In addition, like the corresponding two-dimensional problems both of the loads applied outside the inclusion and inside the inclusion will be treated. Since no general solution has been presented in the literature, the present solutions are checked and verified by the special cases that the laminates as well as the inclusions are composed of the same materials, and that the inclusion is very soft and can be replaced by a hole.

With the Green's functions obtained in this paper, the boundary element for the coupled stretching–bending analysis of holes/cracks/inclusions can be developed, which is then helpful for the study of the interactions between holes, cracks and inclusions, and can also be used for the computation of homogenized elastic constitutive properties of elastic solids with micro-inclusions such as fiber-reinforced composites. Examples of these applications for the two-dimensional analysis can be found in (Hwu and Liao, 1994; Hwu et al., 1995).

2. Stroh-like formalism for coupled stretching–bending analysis

Based upon the Kirchhoff's assumptions for thin plate, the kinematic relations, the constitutive laws and the equilibrium equations for coupled stretching–bending analysis of composite laminates can be written in tensor notation as (Hwu, 2003a)

$$\begin{aligned}
&\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ij} = \frac{1}{2}(\beta_{i,j} + \beta_{j,i}), \\
&N_{ij} = A_{ijkl}\varepsilon_{kl} + B_{ijkl}\kappa_{kl}, \quad M_{ij} = B_{ijkl}\varepsilon_{kl} + D_{ijkl}\kappa_{kl}, \\
&N_{ij,j} = 0, \quad M_{ij,ij} + q = 0, \quad Q_i = M_{ij,j}, \qquad i,j,k,l = 1,2,
\end{aligned} \tag{2.1a}$$

where

$$\beta_1 = -w_{,1}, \quad \beta_2 = -w_{,2}. \tag{2.1b}$$

In the above, the subscript comma stands for differentiation; u_1, u_2 and w are the middle surface displacements in the x_1, x_2 and x_3 directions, and $\beta_i, i = 1,2$, are the negative of the slope of the middle surface; ε_{ij} and κ_{ij} denote the mid-plane strain and plate curvature; N_{ij},M_{ij} and Q_i denote the stress resultants, bending moments and shear forces; A_{ijkl}, B_{ijkl} and D_{ijkl} are, respectively, the extensional, coupling and bending stiffness tensors; q is the lateral distributed load applied on the laminates. Repeated indices imply summation.

Since the Kirchhoff's assumptions are postulated under the condition that transverse shear deformation is neglected, which is usually acceptable for thin elastic plates, the solutions presented in this paper are limited to this kind of plates. For composite laminates with relatively low transverse shear modulus or moderate thickness, transverse shear deformation must be included in Eq. (2.1a) and all the following formulations must be revised.

A general solution satisfying all the basic equations stated in (2.1) has been obtained (Hwu, 2003a) and purposely arranged in the form of Stroh formalism for two-dimensional anisotropic elasticity (Ting, 1996), and hence is called *Stroh-like formalism*. With this formalism, the solution fields of displacements and stresses are expressed as (Hwu, 2003a)

$$\mathbf{u}_d = 2\mathrm{Re}\{\mathbf{A}\mathbf{f}(z)\}, \quad \boldsymbol{\phi}_d = 2\mathrm{Re}\{\mathbf{B}\mathbf{f}(z)\}, \tag{2.2a}$$

where

$$\mathbf{u}_d = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\beta} \end{Bmatrix}, \quad \boldsymbol{\phi}_d = \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \boldsymbol{\beta} = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}, \quad \boldsymbol{\phi} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}, \quad \boldsymbol{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}, \tag{2.2b}$$

and

$$\mathbf{f}(z) = \begin{Bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{Bmatrix}, \quad z_\alpha = x_1 + \mu_\alpha x_2, \qquad \alpha = 1,2,3,4, \tag{2.2c}$$

$$\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4]. \tag{2.2d}$$

Re stands for the real part of a complex number. In $(2.2b)_{5,6}$, ϕ_1, ϕ_2 and ψ_1, ψ_2 are the stress functions related to the stress resultants N_{ij}, shear forces Q_i, effective shear forces V_i and bending moments M_{ij} by

$$\begin{aligned} &N_{i1} = -\phi_{i,2}, \quad N_{i2} = \phi_{i,1}, \\ &M_{i1} = -\psi_{i,2} - \lambda_{i1}\eta, \quad M_{i2} = \psi_{i,1} - \lambda_{i2}\eta, \qquad i = 1,2, \\ &Q_1 = -\eta_{,2}, \quad Q_2 = \eta_{,1}, \quad V_1 = -\psi_{2,22}, \quad V_2 = \psi_{1,11}, \end{aligned} \tag{2.3a}$$

where

$$\eta = \frac{1}{2}\psi_{k,k} = \frac{1}{2}(\psi_{1,1} + \psi_{2,2}), \tag{2.3b}$$

and λ_{ij} is the permutation tensor defined as

$$\lambda_{11} = \lambda_{22} = 0, \quad \lambda_{12} = -\lambda_{21} = 1. \tag{2.3c}$$

$f_\alpha(z_\alpha), \alpha = 1,2,3,4$, are four holomorphic functions of complex variables z_α, which will be determined by the boundary conditions set for each particular problem. μ_α and $(\mathbf{a}_\alpha,\mathbf{b}_\alpha)$ are, respectively, the material eigenvalues and eigenvectors, which can be determined by the following eigen-relation

$$\mathbf{N}\boldsymbol{\xi} = \mu\boldsymbol{\xi}, \tag{2.4a}$$

where $\mathbf{N}$ is a 8×8 real matrix and ξ is a 8×1 column vector defined by

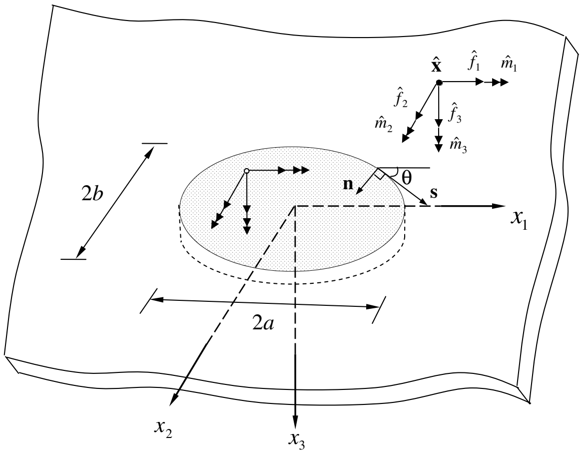

Fig. 1. An elliptic inclusion in laminates subjected to in-plane/out-of-plane concentrated forces and moments.

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \mathbf{N}_2 \\ \mathbf{N}_3 \mathbf{N}_1^T \end{bmatrix}, \quad \xi = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}. \tag{2.4b}$$

The superscript T denotes the transpose of a matrix. The submatrices $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ are the fundamental matrices of elasticity related to the extensional, coupling and bending stiffness tensors. The detailed definitions of $\mathbf{N}_i$ for the coupled stretching–bending problems have been given in (Hwu, 2003a) which are little different from those of two-dimensional problems (Ting, 1996). Moreover, the explicit expressions of $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ as well as their associated eigenvectors **a** and **b** have been found in (Hwu, 2003a; Hsieh and Hwu, 2002).

By using the relations given in (2.3), the stress resultants N_n,N_s,N_{ns}, bending moments M_n,M_s,M_{ns}, shear forces Q_n,Q_s and effective shear forces V_n,V_s in the tangent–normal (s–n) coordinate system, can be obtained directly from the stress functions as follows. The resultant forces $\tilde{t}_i$ and moments $\tilde{m}_i$ along the boundary surfaces, can also be expressed in terms of the stress functions in simple forms (Hsieh and Hwu, 2003; Hwu, 2004).

$$\begin{aligned} &N_n = \mathbf{n}^T \boldsymbol{\phi}_{,s}, \quad N_{ns} = \mathbf{s}^T \boldsymbol{\phi}_{,s} = -\mathbf{n}^T \boldsymbol{\phi}_{,n}, \quad N_s = -\mathbf{s}^T \boldsymbol{\phi}_{,n}, \\ &M_n = \mathbf{n}^T \boldsymbol{\psi}_{,s}, \quad M_{ns} = \mathbf{s}^T \boldsymbol{\psi}_{,s} - \eta = -\mathbf{n}^T \boldsymbol{\psi}_{,n} + \eta, \quad M_s = -\mathbf{s}^T \boldsymbol{\psi}_{,n}, \\ &Q_n = \eta_{,s}, Q_s = -\eta_{,n}, \quad V_n = (\mathbf{s}^T \boldsymbol{\psi}_{,s})_{,s}, \quad V_s = -(\mathbf{n}^T \boldsymbol{\psi}_{,n})_{,n}, \end{aligned} \tag{2.5a}$$

where

$$\begin{aligned} &\eta = \frac{1}{2}(\mathbf{s}^T \boldsymbol{\psi}_{,s} + \mathbf{n}^T \boldsymbol{\psi}_{,n}), \\ &\mathbf{s}^T = (\cos\theta, \sin\theta), \quad \mathbf{n}^T = (-\sin\theta, \cos\theta), \end{aligned} \tag{2.5b}$$

and θ is the angle directed clockwise from the positive x_1-axis to the tangential direction **s** (Fig. 1). Note that the material eigenvalues μ_k have been assumed to be distinct in general solution (2.2). Moreover, the four pairs of material eigenvectors ($\mathbf{a}_k$,$\mathbf{b}_k$), $k = 1,2,3,4$, are assumed to be those corresponding to the eigenvalues with positive imaginary parts. For the laminates whose eigenvalues are repeated, generally called *degenerate laminates*, small perturbation in their values may be introduced to avoid degeneracy (Hwu and Yen, 1991), or higher order eigenvectors may be introduced to get solutions analytically (Yin, 2003a,b, 2005a,b).

3. Boundary conditions

Consider an infinite composite laminate containing an elliptical inclusion subjected to a concentrated force $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and moment $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ at point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ (Fig. 1). The contour of the inclusion boundary is represented by

$$x_1 = a\cos\omega, \quad x_2 = b\sin\omega, \tag{3.1}$$

where $2a$, $2b$ are the major and minor axes of the ellipse and ω is a real parameter. The inclusion and the matrix (the laminates) are assumed to be perfectly bonded along the interface, and hence the displacements and surface tractions across the interface should be continuous.

To describe the above mentioned problem, two boundary conditions in terms of the generalized displacement and stress function vectors, $\mathbf{u}_d$ and $\boldsymbol{\phi}_d$, can be written as follows.

3.1. Concentrated forces and moments

Following the detailed derivation presented in (Hwu, 2004), we see that the relations between the resultant forces/moments and the stress functions are different for different loading directions, in-plane or out-of-plane. Thus, for the convenience of discussion, the associated boundary conditions of concentrated forces and moments are suggested to be presented in three different loading conditions, i.e., (1) $\hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2$; (2) $\hat{f}_3$; (3) $\hat{m}_3$ (Becker, 1995; Hwu, 2004). Considering the force and moment equilibrium and single-valued displacement conditions, the boundary conditions stating the concentrated forces and moments can be expressed in terms of the generalized displacement and stress function vectors, $\mathbf{u}_d$ and $\boldsymbol{\phi}_d$ as follows (Hwu, 2004). For the sake of brevity in this paper some dependent and unnecessary relations are ignored and only the final simplified relations are presented.

$$\text{Case 1. } \hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2 : \oint_C d\boldsymbol{\phi}_d = \hat{\mathbf{p}}, \quad \oint_C d\mathbf{u}_d = \mathbf{0}, \text{ around the point } \hat{\mathbf{x}}. \tag{3.2}$$

$$\text{Case 2. } \hat{f}_3 : \oint_C d\eta = \hat{f}_3, \quad \oint_C d\mathbf{u}_{d,1} = \mathbf{0}, \quad \oint_C d\mathbf{u}_{d,2} = \mathbf{0}, \text{ around the point } \hat{\mathbf{x}}. \tag{3.3}$$

$$\begin{aligned} &\text{Case 3. } \hat{m}_3 : \oint_C d\Phi = -\hat{m}_3, \quad \oint_C dw = 0, \text{ around the point } \hat{\mathbf{x}}, \\ &\qquad \mathbf{n}^T\boldsymbol{\phi} = \mathbf{n}^T\boldsymbol{\psi} = \mathbf{s}^T\boldsymbol{\psi}_{,s} = 0, \text{ along any arbitrary surface boundary.} \end{aligned} \tag{3.4}$$

In Eqs. (3.2) and (3.4),

$$\hat{\mathbf{p}} = (\hat{f}_1 \quad \hat{f}_2 \quad \hat{m}_2 \quad -\hat{m}_1)^T, \quad \Phi_{,1} = \phi_2, \quad \Phi_{,2} = -\phi_1. \tag{3.5}$$

Note that through the definitions of rotation angles β_i and generalized displacement vector $\mathbf{u}_d$ given in (2.1b) and (2.2b), we see that the conditions stated in (3.3) are still not all independent. They are related by

$$\mathbf{i}_3^T\mathbf{u}_{d,2} = \mathbf{i}_4^T\mathbf{u}_{d,1}, \tag{3.6a}$$

where $\mathbf{i}_k$, $k = 1,2,3,4$, are the unit base vectors defined as

$$\mathbf{i}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{i}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{i}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{i}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{3.6b}$$

3.2. Displacement and traction continuity

If the inclusion and the matrix are assumed to be perfectly bonded along the interface, the displacements and surface tractions across the interface should be continuous and the associated boundary conditions can be written as

$$\left.\begin{aligned} &u_n^{(1)} = u_n^{(2)}, \quad u_s^{(1)} = u_s^{(2)}, \quad \beta_n^{(1)} = \beta_n^{(2)}, \quad w_n^{(1)} = w_n^{(2)}, \\ &N_n^{(1)} = N_n^{(2)}, \quad N_{ns}^{(1)} = N_{ns}^{(2)}, \quad M_n^{(1)} = M_n^{(2)}, \quad V_n^{(1)} = V_n^{(2)}, \end{aligned}\right. \text{along the interface,} \tag{3.7}$$

where the superscripts (1) and (2) denote, respectively, the quantities of the matrix and the inclusion. Because in Stroh-like formalism the solution fields are expressed in terms of the generalized displacement and stress function vectors, to employ this formalism it is better to rewrite (3.7) in terms of $\mathbf{u}_d$ and $\boldsymbol{\phi}_d$. With this understanding, by the relations (2.1b), (2.2b) and (2.5a) the displacement and traction continuity conditions can be rewritten as

$$\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}, \quad \boldsymbol{\phi}_d^{(1)} = \boldsymbol{\phi}_d^{(2)}, \quad \text{along the interface.} \tag{3.8}$$

4. General solutions

Since the shape of the inclusion is ellipse, without any special treatment it is difficult to find a proper function satisfying the boundary conditions along the elliptical interface. Through the experiences of the related two-dimensional problems (Hwu and Yen, 1993) and hole coupling problems (Hwu, 2005), we know that the best way is expressing the general solutions in terms of the transformed complex variable ζ_α instead of z_α, which will transform the elliptical boundary into a unit circle $|\zeta| = 1$. The relation between z_α and ζ_α is

$$z_\alpha = \frac{1}{2}\left\{(a - \mathrm{i}b\mu_\alpha)\zeta_\alpha + (a + \mathrm{i}b\mu_\alpha)\frac{1}{\zeta_\alpha}\right\}, \quad \alpha = 1,2,3,4, \tag{4.1a}$$

or inversely

$$\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - b^2\mu_\alpha^2}}{a - \mathrm{i}b\mu_\alpha}, \quad \alpha = 1,2,3,4. \tag{4.1b}$$

Substituting (3.1) and $(2.2c)_2$ into (4.1b), we have

$$\zeta_\alpha = \cos\omega + \mathrm{i}\sin\omega = \mathrm{e}^{\mathrm{i}\omega} = \sigma, \quad \text{along the elliptical interface.} \tag{4.2}$$

Using the method of analytical continuation and understanding that the unknown complex function vector $\mathbf{f}(z)$ is better expressed in terms of the arguments ζ_α, the general solution (2.2) for the matrix and inclusion can now be written as

$$\begin{aligned} \mathbf{u}_d^{(1)} &= 2\mathrm{Re}\{\mathbf{A}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, \quad & \boldsymbol{\phi}_d^{(1)} &= 2\mathrm{Re}\{\mathbf{B}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, \\ \mathbf{u}_d^{(2)} &= 2\mathrm{Re}\{\mathbf{A}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\}, \quad & \boldsymbol{\phi}_d^{(2)} &= 2\mathrm{Re}\{\mathbf{B}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\}, \end{aligned} \tag{4.3}$$

where the subscripts 1 and 2 denote, respectively, the matrix and the inclusion. ζ_α^* is the mapped point of $z_\alpha^* = x_1 + \mu_\alpha^* x_2$ where μ_α^* is the material eigenvalue of the inclusion. $\mathbf{f}_0$ is a function associated with the unperturbed elastic field and is holomorphic in the entire domain except some singular points. $\mathbf{f}_1$ and $\mathbf{f}_2$ are functions corresponding to the perturbed field of the matrix and the inclusion, and are holomorphic, respectively, in the regions of matrix (S_1) and inclusion (S_2) (see Fig. 2).

As described in (Hwu and Yen, 1993) and shown in Fig. 2 that the transformation (4.1) will map the points outside the elliptic inclusion into the points outside the unit circle in ζ_α-domain. Whereas the points inside the elliptic inclusion will be mapped into an annular ring of $\sqrt{m_\alpha} \leqslant |\zeta_\alpha| \leqslant 1$ and is a one-to-one transformation only when the following restriction is satisfied

$$f(\sqrt{\gamma_\alpha^*}\sigma) = f(\sqrt{\gamma_\alpha^*}/\sigma), \tag{4.4a}$$

where

$$\gamma_\alpha^* = \frac{a + \mathrm{i}b\mu_\alpha^*}{a - \mathrm{i}b\mu_\alpha^*} = m_\alpha \mathrm{e}^{2\mathrm{i}\theta_\alpha}. \tag{4.4b}$$

Upon the requirement of (4.4), the function vector $\mathbf{f}_2\,(\zeta^*)$ which is holomorphic in S_2 may now be written as

$$\mathbf{f}_2(\zeta^*) = \mathbf{f}_2^+(\zeta^*) + \mathbf{f}_2^-(\zeta^*), \tag{4.5a}$$

where $\mathbf{f}_2^+(\zeta^*)$ and $\mathbf{f}_2^-(\zeta^*)$ are holomorphic in S_1 and $S_2 + S_0$, respectively, and are related by

$$\mathbf{f}_2^+(\zeta^*) = \mathbf{g}_2(\sqrt{\gamma_\alpha^*}/\zeta^*), \quad \mathbf{f}_2^-(\zeta^*) = \mathbf{g}_2(\zeta^*/\sqrt{\gamma_\alpha^*}), \tag{4.5b}$$

in which $\mathbf{g}_2$ is a function vector to be determined through the satisfaction of the boundary conditions.

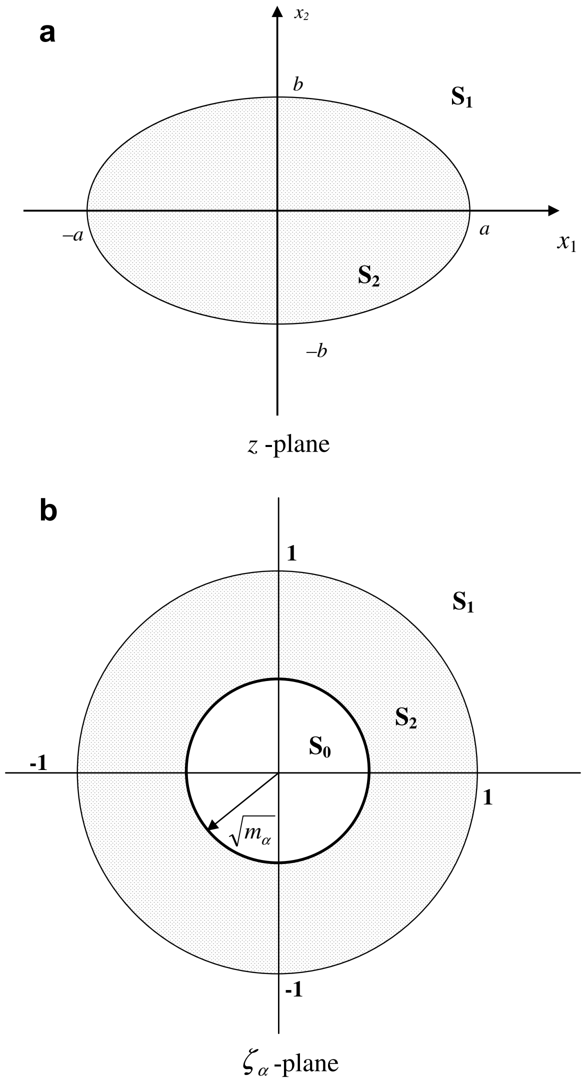

Fig. 2. Mapping from z-plane to ζ_α-plane.

As discussed in (Hwu, 2005), the function vectors $\mathbf{f}_0(\zeta)$ and $\mathbf{f}_0^*(\zeta^*)$ are usually chosen to be the one corresponding to the unperturbed elastic field, i.e., the one for the composite laminates without inclusions. According to the holomorphic characteristics, they may be split into three parts, i.e.,

$$\mathbf{f}_0(\zeta) = \mathbf{f}_0^+(\zeta) + \mathbf{f}_0^-(\zeta) + \mathbf{f}_s(\zeta), \tag{4.6a}$$

$$\mathbf{f}_0^*(\zeta^*) = \mathbf{f}_0^{*+}(\zeta^*) + \mathbf{f}_0^{*-}(\zeta^*) + \mathbf{f}_s^*(\zeta^*), \tag{4.6b}$$

where $\mathbf{f}_0^+(\zeta)$ and $\mathbf{f}_0^{*+}(\zeta^*)$ are holomorphic in S_1, $\mathbf{f}_0^-(\zeta)$ and $\mathbf{f}_0^{*-}(\zeta^*)$ are holomorphic in $S_2 + S_0$, and $\mathbf{f}_s(\zeta)$ and $\mathbf{f}_s^*(\zeta^*)$ are singular function vectors and are not holomorphic in either S_1 or $S_2 + S_0$.

From the experience of two-dimensional problems (Hwu and Yen, 1993), we know that the method of analytical continuation is a powerful method for finding the solutions satisfying all the boundary conditions. To employ this method, the following property is usually used, i.e., if $f(\zeta)$ is holomorphic outside the unit circle S^+ then $\overline{f(1/\bar{\zeta})}$ will be holomorphic inside the unit circle S^-, and vice versa. Using this property and the holomorphic conditions discussed following (4.3), we now list the holomorphic condition of each function vector in Table 1.

Table 1
Holomorphic condition of each function vector

Region	Holomorphic function						
S_1	$\underline{\mathbf{f}_1(\zeta)}$	$\underline{\mathbf{f}_2^+(\zeta^*)}$	$\overline{\mathbf{f}_2^-(1/\bar{\zeta}^*)}$	$\underline{\mathbf{f}_0^+(\zeta)}$	$\overline{\mathbf{f}_0^-(1/\bar{\zeta})}$	$\underline{\mathbf{f}_0^{*+}(\zeta^*)}$	$\overline{\mathbf{f}_0^{*-}(1/\bar{\zeta}^*)}$
S_2+S_0	$\mathbf{f}_1(1/\bar{\zeta})$	$\mathbf{f}_2^+(1/\bar{\zeta}^*)$	$\mathbf{f}_2^-(\zeta^*)$	$\mathbf{f}_0^+(1/\bar{\zeta})$	$\mathbf{f}_0^-(\zeta)$	$\mathbf{f}_0^{*+}(1/\bar{\zeta}^*)$	$\mathbf{f}_0^{*-}(\zeta^*)$
S_2	$\mathbf{f}_2(\zeta^*)$						

5. Forces/moments outside the inclusions

If the in-plane/out-of-plane concentrated forces and moments are located outside the inclusion,

$$\mathbf{f}_0^*(\zeta^*) = \mathbf{0}, \tag{5.1}$$

and $\mathbf{f}_0(\zeta)$ can be selected to be the one associated with the solutions for the composite laminates without inclusions, which has been found in (Hwu, 2004) and will satisfy the boundary conditions shown in (3.2)–(3.4) depending the loading type. They are

$$\text{Case 1. } \mathbf{f}_0(z) = \frac{1}{2\pi i} < \log(z_\alpha - \hat{z}_\alpha) > \mathbf{A}_1^T\hat{\mathbf{p}}; \tag{5.2a}$$

$$\text{Case 2. } \mathbf{f}_0(z) = \frac{\hat{f}_3}{2\pi i} < (z_\alpha - \hat{z}_\alpha)[\log(z_\alpha - \hat{z}_\alpha) - 1] > \mathbf{A}_1^T\mathbf{i}_3; \tag{5.2b}$$

$$\text{Case 3. } \mathbf{f}_0(z) = \frac{\hat{m}_3}{2\pi i} < \frac{1}{z_\alpha - \hat{z}_\alpha} > \mathbf{A}_1^T\mathbf{i}_2. \tag{5.2c}$$

In the above, the angular bracket stands for the diagonal matrix whose components vary according to the subscript α, $\alpha = 1,2,3,4$, i.e., $<f_\alpha> = \mathrm{diag}[f_1, f_2, f_3, f_4]$.

The solutions shown in (5.2) are expressed in terms of z_α instead of ζ_α, which are not convenient for considering the continuity conditions along the elliptic interface. Moreover, to employ the method of analytical continuation, it is necessary for us to split $\mathbf{f}_0(z)$ into three parts as those shown in (4.6a). This work has been done in deriving the Green's functions for holes/cracks in laminates (Hwu, 2005). Thus, no detailed derivation will be presented here and only the results of $\mathbf{f}_0^-(\zeta)$ are shown below.

$$\text{Case 1. } \mathbf{f}_0^-(\zeta) = \frac{1}{2\pi i} < \log(\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{A}_1^T\hat{\mathbf{p}} \tag{5.3a}$$

$$\text{Case 2. } \mathbf{f}_0^-(\zeta) = \frac{\hat{f}_3}{2\pi i} < (z_\alpha - \hat{z}_\alpha)\log(\zeta_\alpha - \hat{\zeta}_\alpha) + c_{2\alpha}(\zeta_\alpha - \hat{\zeta}_\alpha) - c_{3\alpha}(\zeta_\alpha^{-1} - \hat{\zeta}_\alpha^{-1}) > \mathbf{A}_1^T\mathbf{i}_3 \tag{5.3b}$$

$$\text{Case 3. } \mathbf{f}_0^-(\zeta) = \frac{\hat{m}_3}{2\pi i} < \frac{c_{4\alpha}\hat{\zeta}_\alpha}{\zeta_\alpha - \hat{\zeta}_\alpha} > \mathbf{A}_1^T\mathbf{i}_2 \tag{5.3c}$$

where

$$\begin{aligned} c_\alpha &= \frac{1}{2}(a - ib\mu_\alpha), \quad c_{2\alpha} = c_\alpha(\log c_\alpha - 1), \\ c_{3\alpha} &= c_\alpha\gamma_\alpha\log(-\hat{\zeta}_\alpha), \quad c_{4\alpha} = \frac{1}{c_\alpha(\hat{\zeta}_\alpha - \gamma_\alpha/\hat{\zeta}_\alpha)} \end{aligned} \tag{5.4}$$

Note that in the above $\mathbf{f}_s(\zeta)$ and $\mathbf{f}_0^+(\zeta)$ are not shown because no singular function vector remained for the present problem, i.e.,

$$\mathbf{f}_s(\zeta) = \mathbf{0}, \tag{5.5}$$

and $\mathbf{f}_0^+(\zeta)$ has the same holomorphic condition as $\mathbf{f}_1(\zeta)$ (see Table 1) and can be combined together, i.e., we may let

$$\breve{\mathbf{f}}_1(\zeta) = \mathbf{f}_1(\zeta) + \mathbf{f}_0^+(\zeta). \tag{5.6}$$

Substituting (4.5a), (4.6a), (5.1), (5.5) and (5.6) into (4.3), the generalized displacement and stress function vectors can now be rewritten as

$$
\begin{aligned}
\mathbf{u}_d^{(1)} &= \mathbf{A}_1[\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta)] + \overline{\mathbf{A}}_1[\overline{\mathbf{f}_0^-(\zeta)} + \overline{\breve{\mathbf{f}}_1(\zeta)}],\\
\boldsymbol{\phi}_d^{(1)} &= \mathbf{B}_1[\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta)] + \overline{\mathbf{B}}_1[\overline{\mathbf{f}_0^-(\zeta)} + \overline{\breve{\mathbf{f}}_1(\zeta)}],\\
\mathbf{u}_d^{(2)} &= \mathbf{A}_2[\mathbf{f}_2^+(\zeta^*) + \mathbf{f}_2^-(\zeta^*)] + \overline{\mathbf{A}}_2[\overline{\mathbf{f}_2^+(\zeta^*)} + \overline{\mathbf{f}_2^-(\zeta^*)}],\\
\boldsymbol{\phi}_d^{(2)} &= \mathbf{B}_2[\mathbf{f}_2^+(\zeta^*) + \mathbf{f}_2^-(\zeta^*)] + \overline{\mathbf{B}}_2[\overline{\mathbf{f}_2^+(\zeta^*)} + \overline{\mathbf{f}_2^-(\zeta^*)}].
\end{aligned}
\tag{5.7}
$$

Since $\mathbf{f}_0(z)$ shown in (5.2) have satisfied the boundary conditions for the concentrated forces and moments, to determine the unknown function vectors $\breve{\mathbf{f}}_1(\zeta), \mathbf{f}_2^+(\zeta^*)$ and $\mathbf{f}_2^-(\zeta^*)$ we now consider the displacement and traction continuity conditions shown in (3.8). Substituting (5.7) into (3.8) with $\zeta_\alpha = \sigma$ along the interface, and following the standard approach of analytical continuation (Hwu and Yen, 1993) with the knowledge of the holomorphic conditions shown in Table 1, we can obtain

$$
\begin{aligned}
&\left.\begin{aligned}
\mathbf{A}_1\breve{\mathbf{f}}_1(\zeta) + \overline{\mathbf{A}}_1\overline{\mathbf{f}_0^-(1/\bar{\zeta})} &= \mathbf{A}_2\mathbf{f}_2^+(\zeta) + \overline{\mathbf{A}}_2\overline{\mathbf{f}_2^-(1/\bar{\zeta})}\\
\mathbf{B}_1\breve{\mathbf{f}}_1(\zeta) + \overline{\mathbf{B}}_1\overline{\mathbf{f}_0^-(1/\bar{\zeta})} &= \mathbf{B}_2\mathbf{f}_2^+(\zeta) + \overline{\mathbf{B}}_2\overline{\mathbf{f}_2^-(1/\bar{\zeta})}
\end{aligned}\right\}, \zeta \in S_1,\\
&\left.\begin{aligned}
\mathbf{A}_1\mathbf{f}_0^-(\zeta) + \overline{\mathbf{A}}_1\overline{\breve{\mathbf{f}}_1(1/\bar{\zeta})} &= \mathbf{A}_2\mathbf{f}_2^-(\zeta) + \overline{\mathbf{A}}_2\overline{\mathbf{f}_2^+(1/\bar{\zeta})}\\
\mathbf{B}_1\mathbf{f}_0^-(\zeta) + \overline{\mathbf{B}}_1\overline{\breve{\mathbf{f}}_1(1/\bar{\zeta})} &= \mathbf{B}_2\mathbf{f}_2^-(\zeta) + \overline{\mathbf{B}}_2\overline{\mathbf{f}_2^+(1/\bar{\zeta})}
\end{aligned}\right\}, \zeta \in S_0 + S_2.
\end{aligned}
\tag{5.8}
$$

Note that in (5.8) the superscript * denoting the argument of the inclusion has been dropped, that is due to the use of analytical continuation. When employing the method of analytical continuation, the results come up from the continuation across the boundary in which all the arguments $\zeta_\alpha, \zeta_\alpha^*, \alpha = 1,2,3,4$, have the same value $\sigma = e^{i\omega}$. To make the final results have more flexibility to suit for the solution form shown in (2.2c), at this stage the superscript * and/or the subscript α will be dropped, and a replacement of ζ_α or $\zeta_\alpha^*, \alpha = 1,2,3,4$, should be made for each component function of $\mathbf{f}(\zeta)$ at the final stage. To get the complete full field solution possessing the standard form shown in (2.2c), a translating technique may be employed. Detailed explanation about this technique can be found in (Hwu, 1993; Hwu, 2005). This calculation procedure should then be applied throughout this paper without further notification.

Although the solutions of $\mathbf{f}_0^-(\zeta)$ have been given explicitly in (5.3), it is difficult to get the explicit solutions of $\breve{\mathbf{f}}_1(\zeta), \mathbf{f}_2^+(\zeta^*)$ and $\mathbf{f}_2^-(\zeta^*)$ directly from the relations (5.8). Instead they can be obtained in series form. With this consideration, by Taylor's expansion $\mathbf{f}_0^-(\zeta)$ of (5.3) can now be expressed in series form as

$$
\mathbf{f}_0^-(\zeta) = \sum_{k=1}^{\infty} \mathbf{e}_k^- \zeta^k,
\tag{5.9}
$$

where

$$
\text{Case 1. } \mathbf{e}_k^- = \frac{1}{2\pi i} < \frac{-1}{k}\hat{\zeta}_\alpha^{-k} > \mathbf{A}_1^T\hat{\mathbf{p}}
\tag{5.10a}
$$

$$
\begin{aligned}
\text{Case 2. } \mathbf{e}_1^- &= \frac{\hat{f}_3}{2\pi i} < \frac{c_\alpha}{\hat{\zeta}_\alpha}\left[\hat{\zeta}_\alpha \log(-c_\alpha\hat{\zeta}_\alpha) + \frac{\gamma_\alpha}{2\hat{\zeta}_\alpha}\right] > \mathbf{A}_1^T\mathbf{i}_3,\\
\mathbf{e}_k^- &= \frac{\hat{f}_3}{2\pi i} < \frac{c_\alpha}{k\hat{\zeta}_\alpha^k}\left[\frac{-\hat{\zeta}_\alpha}{k-1} + \frac{\gamma_\alpha}{(k+1)\hat{\zeta}_\alpha}\right] > \mathbf{A}_1^T\mathbf{i}_3, \quad k \neq 1
\end{aligned}
\tag{5.10b}
$$

$$
\text{Case 3. } \mathbf{e}_k^- = -\frac{\hat{m}_3}{2\pi i} < c_{4\alpha}\hat{\zeta}_\alpha^{-k} > \mathbf{A}_1^T\mathbf{i}_2
\tag{5.10c}
$$

From the holomorphic condition of $\mathbf{f}_2^+(\zeta)$ and $\mathbf{f}_2^-(\zeta)$, the function $\mathbf{g}_2$ in (4.5b) can also be assumed in series form such as $\mathbf{g}_2(\zeta) = \sum_{k=1}^{\infty}\mathbf{d}_k\zeta^k$. With this series form, the relations given in (4.5b) will provide us the series of $\mathbf{f}_2^+(\zeta)$ and $\mathbf{f}_2^-(\zeta)$. To get a comparable solution with those of two-dimensional problem (Hwu and Yen, 1993), the unknown coefficient $\mathbf{d}_k$ is now replaced by $\mathbf{c}_k = <\gamma_\alpha^{*-k}> \mathbf{d}_k$. With this replacement, we have

$$\mathbf{f}_2^+(\zeta) = \sum_{k=1}^{\infty} <\gamma_\alpha^{*k}> \mathbf{c}_k \zeta^{-k}, \quad \mathbf{f}_2^-(\zeta) = \sum_{k=1}^{\infty} \mathbf{c}_k \zeta^k. \tag{5.11}$$

Adding $\mathbf{f}_2^+(\zeta)$ and $\mathbf{f}_2^-(\zeta)$ together, we get

$$\mathbf{f}_2(\zeta) = \sum_{k=-\infty}^{\infty} \mathbf{c}_k \zeta^k, \tag{5.12a}$$

where

$$\mathbf{c}_{-k} = <\gamma_\alpha^{*k}> \mathbf{c}_k. \tag{5.12b}$$

Note that in the above equations, (5.9)–(5.12), the constant term which associated with rigid body motion has been neglected.

Substituting (5.9) and (5.11) into (5.8), and following the standard approach of analytical continuation (Hwu and Yen, 1993), the coefficient vector $\mathbf{c}_k$ can be obtained as

$$\mathbf{c}_k = \{\mathbf{G}_0 - \overline{\mathbf{G}}_k \overline{\mathbf{G}}_0^{-1} \mathbf{G}_k\}^{-1} \{\mathbf{t}_k - \overline{\mathbf{G}}_k \overline{\mathbf{G}}_0^{-1} \bar{\mathbf{t}}_k\}, \quad k = 1, 2, \ldots, \infty, \tag{5.13a}$$

where

$$\mathbf{G}_0 = \{\overline{\mathbf{M}}_1 + \mathbf{M}_2\}\mathbf{A}_2, \quad \mathbf{G}_k = \{\mathbf{M}_1 - \mathbf{M}_2\}\mathbf{A}_2 <\gamma_\alpha^{*k}>, \quad \mathbf{t}_k = -\mathrm{i}\mathbf{A}_1^{-\mathrm{T}} \mathbf{e}_k^-, \tag{5.13b}$$

and $\mathbf{M}_k$, $k = 1{,}2$, are the impedance matrices defined as

$$\mathbf{M}_1 = -\mathrm{i}\mathbf{B}_1\mathbf{A}_1^{-1}, \quad \mathbf{M}_2 = -\mathrm{i}\mathbf{B}_2\mathbf{A}_2^{-1}. \tag{5.13c}$$

Having the solution of $\mathbf{c}_k$, function vectors $\mathbf{f}_2^+(\zeta), \mathbf{f}_2^-(\zeta)$ and $\mathbf{f}_2(\zeta)$ are determined by (5.11) and (5.12). Function vector $\breve{\mathbf{f}}_1(\zeta)$ can then be obtained either from the first or the second equation of (5.8). The results are

$$\breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} \mathbf{A}_1^{-1} \{\overline{\mathbf{A}}_1 \bar{\mathbf{e}}_k^- - \mathbf{A}_2 <\gamma_\alpha^{*k}> \mathbf{c}_k - \overline{\mathbf{A}}_2 \bar{\mathbf{c}}_k\} \zeta^{-k}, \tag{5.14a}$$

or

$$\breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} \mathbf{B}_1^{-1} \{\overline{\mathbf{B}}_1 \bar{\mathbf{e}}_k^- - \mathbf{B}_2 <\gamma_\alpha^{*k}> \mathbf{c}_k - \overline{\mathbf{B}}_2 \bar{\mathbf{c}}_k\} \zeta^{-k}. \tag{5.14b}$$

From the above derivation we see that to obtain the solutions of generalized displacement and stress function vectors from (5.7), we need to combine the results of (5.3) for $\mathbf{f}_0^-(\zeta)$, (5.12) and (5.13) for $\mathbf{f}_2(\zeta^*)$, and (5.14) for $\breve{\mathbf{f}}_1(\zeta)$. There is no need to calculate the separate solutions for $\mathbf{f}_0^+(\zeta), \mathbf{f}_2^+(\zeta^*)$ and $\mathbf{f}_2^-(\zeta^*)$, unless special purpose is requested. Table 2 shows the comparison of the present solutions with those of the corresponding two-dimensional problems. From this comparison, we observe that the solutions obtained here for Case 1 possess exactly the same mathematical form as those obtained in (Hwu and Yen, 1993) for the corresponding two-dimensional problems. The only difference is the contents of the symbols. While for the other two Cases, one more difference comes from $\mathbf{f}_0^-(\zeta)$. Note that as stated in the paragraph following Eq. (5.8), to get the complete full field solutions the translating technique (Hwu, 1993) has been employed in Table 2.

Instead of the full field solutions, in engineering application one is usually interested in the stress resultants and moments along the interface boundary. From the relations shown in (2.5), we know that the calculation of stress resultants and moments replies upon the calculation of the differentials $\boldsymbol{\phi}_{d,s}$ and $\boldsymbol{\phi}_{d,n}$. Due to the continuity conditions (3.8), the derivative of $\boldsymbol{\phi}_d$ along the interface should be continuous across the interface, i.e., $\boldsymbol{\phi}_{d,s}^{(1)} = \boldsymbol{\phi}_{d,s}^{(2)}$. On the other hand, $\boldsymbol{\phi}_{d,n}^{(1)}$ may not equal to $\boldsymbol{\phi}_{d,n}^{(2)}$. Thus, to calculate $\boldsymbol{\phi}_{d,s}$ along the interface one may use the field solutions of the inclusion or the matrix. While for the calculation of $\boldsymbol{\phi}_{d,n}$, we may differentiate $\boldsymbol{\phi}_d$ directly with respect to the normal direction $\mathbf{n}$ for both the inclusion and the matrix, or calculate indirectly from $\boldsymbol{\phi}_{d,s}$ through the following relation (Ting and Yan, 1991; Hwu and Yen, 1993).

Table 2
Green's functions for inclusion problems – loads outside the inclusions

Two-dimensional problems: coupled inplane–antiplane deformations (Hwu and Yen, 1993) point load: $\hat{f}_1, \hat{f}_2, \hat{f}_3$

$$\mathbf{u}_1 = 2\mathrm{Re}\{\mathbf{A}_1[\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta)]\}, \quad \boldsymbol{\phi}_1 = 2\mathrm{Re}\{\mathbf{B}_1[\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta)]\},$$
$$\mathbf{u}_2 = 2\mathrm{Re}\{\mathbf{A}_2\mathbf{f}_2(\zeta^*)\}, \quad \boldsymbol{\phi}_2 = 2\mathrm{Re}\{\mathbf{B}_2\mathbf{f}_2(\zeta^*)\},$$
$$\mathbf{u}_1, \mathbf{u}_2 : 3\times 1, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 : 3\times 1, \mathbf{A}_1, \mathbf{A}_2 : 3\times 3, \quad \mathbf{B}_1, \mathbf{B}_2 : 3\times 3; \quad \mathbf{f}_0^-(\zeta), \breve{\mathbf{f}}_1(\zeta), \mathbf{f}_2(\zeta^*) : 3\times 1$$

$$\mathbf{f}_0^-(\zeta) = \frac{1}{2\pi i} < \log(\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{A}_1^{\mathrm{T}}\hat{\mathbf{p}}$$
$$\breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{A}_1^{-1}\{\overline{\mathbf{A}}_1\overline{\mathbf{e}}_k^- - \mathbf{A}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k - \overline{\mathbf{A}}_2\overline{\mathbf{c}}_k\}, \text{ or } \breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{B}_1^{-1}\{\overline{\mathbf{B}}_1\overline{\mathbf{e}}_k^- - \mathbf{B}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k - \overline{\mathbf{B}}_2\overline{\mathbf{c}}_k\}$$
$$\mathbf{f}_2(\zeta^*) = \sum_{k=-\infty}^{\infty} < \zeta_\alpha^{*k} > \mathbf{c}_k, \mathbf{c}_{-k} = < \gamma_\alpha^{*k} > \mathbf{c}_k$$

in which

$$\mathbf{e}_k^- = \tfrac{1}{2\pi i} < \tfrac{-1}{k}\hat{\zeta}_\alpha^{-k} > \mathbf{A}_1^{\mathrm{T}}\hat{\mathbf{p}}$$

Coupled stretching–bending problems Case 1: $\hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2$; Case 2: $\hat{f}_3$; Case 3: $\hat{m}_3$

$$\mathbf{u}_d^{(1)} = 2\mathrm{Re}\{\mathbf{A}_1[\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta)]\}, \quad \boldsymbol{\phi}_d^{(1)} = 2\mathrm{Re}\{\mathbf{B}_1[\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta)]\},$$
$$\mathbf{u}_d^{(2)} = 2\mathrm{Re}\{\mathbf{A}_2\mathbf{f}_2(\zeta^*)\}, \quad \boldsymbol{\phi}_d^{(2)} = 2\mathrm{Re}\{\mathbf{B}_2\mathbf{f}_2(\zeta^*)\},$$
$$\mathbf{u}_d^{(1)}, \mathbf{u}_d^{(2)} : 4\times 1, \quad \boldsymbol{\phi}_d^{(1)}, \boldsymbol{\phi}_d^{(2)} : 4\times 1, \mathbf{A}_1, \mathbf{A}_2 : 4\times 4, \quad \mathbf{B}_1, \mathbf{B}_2 : 4\times 4; \quad \mathbf{f}_0^-(\zeta), \breve{\mathbf{f}}_1(\zeta), \mathbf{f}_2(\zeta^*) : 4\times 1$$

Case 1: $\mathbf{f}_0^-(\zeta) = \dfrac{1}{2\pi i} < \log(\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{A}_1^{\mathrm{T}}\hat{\mathbf{p}}$

$$\breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{A}_1^{-1}\{\overline{\mathbf{A}}_1\overline{\mathbf{e}}_k^- - \mathbf{A}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k - \overline{\mathbf{A}}_2\overline{\mathbf{c}}_k\}$$
$$\text{or} \quad \breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{B}_1^{-1}\{\overline{\mathbf{B}}_1\overline{\mathbf{e}}_k^- - \mathbf{B}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k - \overline{\mathbf{B}}_2\overline{\mathbf{c}}_k\}$$
$$\mathbf{f}_2(\zeta^*) = \sum_{k=-\infty}^{\infty} < \zeta_\alpha^{*k} > \mathbf{c}_k, \mathbf{c}_{-k} = < \gamma_\alpha^{*k} > \mathbf{c}_k$$

Case 2: $\mathbf{f}_0^-(\zeta) = \dfrac{\hat{f}_3}{2\pi i} < (z_\alpha - \hat{z}_\alpha)\log(\zeta_\alpha - \hat{\zeta}_\alpha) + c_{2\alpha}(\zeta_\alpha - \hat{\zeta}_\alpha) - c_{3\alpha}(\zeta_\alpha^{-1} - \hat{\zeta}_\alpha^{-1}) > \mathbf{A}_1^{\mathrm{T}}\mathbf{i}_3$

$$\breve{\mathbf{f}}_1(\zeta), \quad \mathbf{f}_2(\zeta^*) : \text{same mathematical form as Case 1, the only difference is the content of } \mathbf{e}_k^- \text{ and } \mathbf{c}_k$$

Case 3: $\mathbf{f}_0^-(\zeta) = \dfrac{\hat{m}_3}{2\pi i} < \dfrac{c_{4\alpha}\hat{\zeta}_\alpha}{\zeta_\alpha - \hat{\zeta}_\alpha} > \mathbf{A}_1^{\mathrm{T}}\mathbf{i}_2$

$$\breve{\mathbf{f}}_1(\zeta), \quad \mathbf{f}_2(\zeta^*) : \text{same mathematical form as Case 1, the only difference is the content of } \mathbf{e}_k^- \text{ and } \mathbf{c}_k$$

in which

Case 1: $\mathbf{e}_k^- = \frac{1}{2\pi i} < \frac{-1}{k}\hat{\zeta}_\alpha^{-k} > \mathbf{A}_1^{\mathrm{T}}\hat{\mathbf{p}}$

Case 2: $\mathbf{e}_1^- = \dfrac{\hat{f}_3}{2\pi i} < \dfrac{c_\alpha}{\hat{\zeta}_\alpha}\left[\hat{\zeta}_\alpha\log(-c_\alpha\hat{\zeta}_\alpha) + \dfrac{\gamma_\alpha}{2\hat{\zeta}_\alpha}\right] > \mathbf{A}_1^{\mathrm{T}}\mathbf{i}_3,$

$$\mathbf{e}_k^- = \frac{\hat{f}_3}{2\pi i} < \frac{c_\alpha}{k\hat{\zeta}_\alpha^k}\left[\frac{-\hat{\zeta}_\alpha}{k-1} + \frac{\gamma_\alpha}{(k+1)\hat{\zeta}_\alpha}\right] > \mathbf{A}_1^{\mathrm{T}}\mathbf{i}_3, k \neq 1$$

Case 3: $\mathbf{e}_k^- = -\frac{\hat{m}_3}{2\pi i} < c_{4\alpha}\hat{\zeta}_\alpha^{-k} > \mathbf{A}_1^{\mathrm{T}}\mathbf{i}_2$

$\mathbf{c}_k = \{\mathbf{G}_0 - \overline{\mathbf{G}}_k\overline{\mathbf{G}}_0^{-1}\mathbf{G}_k\}^{-1}\{\mathbf{t}_k - \overline{\mathbf{G}}_k\overline{\mathbf{G}}_0^{-1}\overline{\mathbf{t}}_k\}, k = 1, 2, \ldots, \infty;$
$\mathbf{G}_0 = \{\overline{\mathbf{M}}_1 + \mathbf{M}_2\}\mathbf{A}_2, \ \mathbf{G}_k = \{\mathbf{M}_1 - \mathbf{M}_2\}\mathbf{A}_2 < \gamma_\alpha^{*k} >, \mathbf{t}_k = -i\mathbf{A}_1^{-\mathrm{T}}\mathbf{e}_k^-; \ \mathbf{M}_1 = -i\mathbf{B}_1\mathbf{A}_1^{-1}, \mathbf{M}_2 = -i\mathbf{B}_2\mathbf{A}_2^{-1}$
Refer to Table 4 for the other symbols.

$$\begin{Bmatrix} \mathbf{u}_{d,n} \\ \boldsymbol{\phi}_{d,n} \end{Bmatrix} = \mathbf{N}(\theta) \begin{Bmatrix} \mathbf{u}_{d,s} \\ \boldsymbol{\phi}_{d,s} \end{Bmatrix}, \tag{5.15}$$

where $\mathbf{N}(\theta)$ is the generalized matrix of the fundamental matrix $\mathbf{N}$(Hsieh and Hwu, 2003). By using chain rule, two relations useful for the calculation of $\boldsymbol{\phi}_{d,s}$ and $\boldsymbol{\phi}_{d,n}$ have been obtained as (Hwu, 2005)

$$\frac{\partial f}{\partial n} = \frac{\mathrm{i}e^{\mathrm{i}\omega}\mu_\alpha(\theta)}{\rho}\frac{\partial f}{\partial \zeta_\alpha}, \quad \frac{\partial f}{\partial s} = \frac{\mathrm{i}e^{\mathrm{i}\omega}}{\rho}\frac{\partial f}{\partial \zeta_\alpha}, \quad \text{along the interface boundary,} \tag{5.16}$$

where $\mu_\alpha(\theta)$ is the generalized material eigenvalue related to μ_α by

$$\mu_\alpha(\theta) = \frac{-\sin\theta + \mu_\alpha\cos\theta}{\cos\theta + \mu_\alpha\sin\theta}. \tag{5.17}$$

Using the second relation of (5.16) and the solutions for the inclusion shown in Table 2, we obtain

$$\boldsymbol{\phi}_{d,s} = \boldsymbol{\phi}_{d,s}^{(1)} = \boldsymbol{\phi}_{d,s}^{(2)} = 2\mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \frac{\mathrm{i}k e^{\mathrm{i}k\omega}}{\rho}\mathbf{B}_2\mathbf{c}_k\right\}, \quad \text{along the interface boundary.} \tag{5.18}$$

With the solution (5.18) and the relation (5.15), $\boldsymbol{\phi}_{d,n}$ can be obtained for both the inclusion and the matrix. Or alternatively, we can obtain $\boldsymbol{\phi}_{d,n}$ directly through the use of $(5.16)_1$ and the solutions for the matrix and inclusion shown in Table 2.

6. Forces/moments inside the inclusions

The holomorphic condition discussed in Section 4 will change if the singular point $\hat{\zeta}_\alpha$ is located inside the inclusion. For example, function $\log(\zeta_\alpha - \hat{\zeta}_\alpha)$ of (5.3a) which is holomorphic inside the unit circle $(S_2 + S_0)$ when $|\hat{\zeta}_\alpha| > 1$, will not be holomorphic in $S_2 + S_0$ because $\hat{\zeta}_\alpha$ is now located in this region. For this reason, when the concentrated forces and moments are applied on the points inside the inclusion, the solutions obtained in Section 5 become invalid and new solutions should be found. Since the forces/moments are applied on the point $\hat{\zeta}_\alpha^*$ inside the inclusion, instead of $\mathbf{f}_0(\zeta)$ we now select $\mathbf{f}_0^*(\zeta^*)$ to be the one associated with the solutions for the composite laminates without inclusions. That is,

$$\text{Case 1. } \mathbf{f}_0^*(\zeta^*) = \frac{1}{2\pi \mathrm{i}} < \log(z_\alpha^* - \hat{z}_\alpha^*) > \mathbf{A}_2^{\mathrm{T}}\hat{\mathbf{p}}; \tag{6.1a}$$

$$\text{Case 2. } \mathbf{f}_0^*(\zeta^*) = \frac{\hat{f}_3}{2\pi \mathrm{i}} < (z_\alpha^* - \hat{z}_\alpha^*)[\log(z_\alpha^* - \hat{z}_\alpha^*) - 1] > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3; \tag{6.1b}$$

$$\text{Case 3. } \mathbf{f}_0^*(\zeta^*) = \frac{\hat{m}_3}{2\pi \mathrm{i}} < \frac{1}{z_\alpha^* - \hat{z}_\alpha^*} > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_2. \tag{6.1c}$$

Note that the functions shown in (6.1) are written in terms of z_α^*, which can be re-written in terms of ζ_α^* by using the relation given in (4.1a). Also because $\mathbf{f}_0^*(\zeta^*)$ is selected to be a function of argument z_α^*, the restriction (4.4) for one-to-one mapping from z_α^* to ζ_α^* will be satisfied automatically.

Substituting (4.1a) into (6.1a)–(6.1c) and considering the holomorphic condition of each term, we may now split $\mathbf{f}_0^*(\zeta^*)$ into three parts as those shown in (4.6b). They are

Case 1.

$$\begin{aligned} \mathbf{f}_0^{*+}(\zeta^*) &= \frac{1}{2\pi \mathrm{i}} < \log(z_\alpha^* - \hat{z}_\alpha^*) - \log\zeta_\alpha^* > \mathbf{A}_2^{\mathrm{T}}\hat{\mathbf{p}}, \\ \mathbf{f}_0^{*-}(\zeta^*) &= \mathbf{0}, \\ \mathbf{f}_s^*(\zeta^*) &= \frac{1}{2\pi \mathrm{i}} < \log\zeta_\alpha^* > \mathbf{A}_2^{\mathrm{T}}\hat{\mathbf{p}}. \end{aligned} \tag{6.2a}$$

Case 2.

$$\mathbf{f}_0^{*+}(\zeta^*) = \frac{\hat{f}_3}{2\pi i} < (z_\alpha^* - \hat{z}_\alpha^*)\left[\log(z_\alpha^* - \hat{z}_\alpha^*) - \log\zeta_\alpha^* - 1\right] - c_\alpha^*\zeta_\alpha^*(\log c_\alpha^* - 1) > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3,$$
$$\mathbf{f}_0^{*-}(\zeta^*) = \frac{\hat{f}_3}{2\pi i} < c_\alpha^*\zeta_\alpha^*(\log c_\alpha^* - 1) > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3, \tag{6.2b}$$
$$\mathbf{f}_s^*(\zeta^*) = \frac{\hat{f}_3}{2\pi i} < (z_\alpha^* - \hat{z}_\alpha^*)\log\zeta_\alpha^* > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3.$$

Case 3.

$$\mathbf{f}_0^{*+}(\zeta^*) = \frac{\hat{m}_3}{2\pi i} < \frac{1}{z_\alpha^* - \hat{z}_\alpha^*} > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_2,$$
$$\mathbf{f}_0^{*-}(\zeta^*) = \mathbf{0}, \tag{6.2c}$$
$$\mathbf{f}_s^*(\zeta^*) = \mathbf{0}.$$

Unlike the problems considered in Section 5, the function $\mathbf{f}_0(\zeta)$ cannot be set to be zero like that of (5.1) because in region S_1 we need corresponding term $\mathbf{f}_s(\zeta)$ for the singular function $\mathbf{f}_s^*(\zeta^*)$ in region $S_2 + S_0$. Moreover, $\mathbf{f}_0^+(\zeta)$ and $\mathbf{f}_0^-(\zeta)$ are also needed to have comparable terms with $\mathbf{f}_0^{*-}(\zeta)$ which is holomorphic in $S_2 + S_0$ but not holomorphic in S_1. With this consideration, $\mathbf{f}_0(\zeta)$ is also splitted into three parts as that shown in (4.6a), in which each term is assumed as follows.

Case 1. $\mathbf{f}_s(\zeta) =< \log\zeta_\alpha > \mathbf{d}, \quad \mathbf{f}_0^+(\zeta) = \mathbf{f}_0^-(\zeta) = \mathbf{0}.$ (6.3a)

Case 2. $\mathbf{f}_s(\zeta) =< \log\zeta_\alpha > \{< \zeta_\alpha > \mathbf{d}_1 + < \zeta_\alpha^{-1} > \mathbf{d}_{-1} + \mathbf{d}_0\},$

$\mathbf{f}_0^+(\zeta) =< \zeta_\alpha^{-1} > \mathbf{k}_{-1}, \mathbf{f}_0^-(\zeta) =< \zeta_\alpha > \mathbf{k}_1.$ (6.3b)

Case 3. $\mathbf{f}_s(\zeta) = \mathbf{f}_0^+(\zeta) = \mathbf{f}_0^-(\zeta) = \mathbf{0}.$ (6.3c)

Like (5.7), we now substitute (4.5a), (4.6a) and (4.6b) into (4.3) and rewrite the general solutions for the generalized displacement and stress function vectors as

$$\begin{aligned}
\mathbf{u}_d^{(1)} &= \mathbf{A}_1[\mathbf{f}_s(\zeta) + \mathbf{f}_0^+(\zeta) + \mathbf{f}_0^-(\zeta) + \mathbf{f}_1(\zeta)] + \overline{\mathbf{A}}_1[\overline{\mathbf{f}_s(\zeta)} + \overline{\mathbf{f}_0^+(\zeta)} + \overline{\mathbf{f}_0^-(\zeta)} + \overline{\mathbf{f}_1(\zeta)}],\\
\boldsymbol{\phi}_d^{(1)} &= \mathbf{B}_1[\mathbf{f}_s(\zeta) + \mathbf{f}_0^+(\zeta) + \mathbf{f}_0^-(\zeta) + \mathbf{f}_1(\zeta)] + \overline{\mathbf{B}}_1[\overline{\mathbf{f}_s(\zeta)} + \overline{\mathbf{f}_0^+(\zeta)} + \overline{\mathbf{f}_0^-(\zeta)} + \overline{\mathbf{f}_1(\zeta)}],\\
\mathbf{u}_d^{(2)} &= \mathbf{A}_2[\mathbf{f}_s^*(\zeta^*) + \mathbf{f}_0^{*+}(\zeta^*) + \mathbf{f}_0^{*-}(\zeta^*) + \mathbf{f}_2^+(\zeta^*) + \mathbf{f}_2^-(\zeta^*)]\\
&\quad + \overline{\mathbf{A}}_2[\overline{\mathbf{f}_s^*(\zeta^*)} + \overline{\mathbf{f}_0^{*+}(\zeta^*)} + \overline{\mathbf{f}_0^{*-}(\zeta^*)} + \overline{\mathbf{f}_2^+(\zeta^*)} + \overline{\mathbf{f}_2^-(\zeta^*)}],\\
\boldsymbol{\phi}_d^{(2)} &= \mathbf{B}_2[\mathbf{f}_s^*(\zeta^*) + \mathbf{f}_0^{*+}(\zeta^*) + \mathbf{f}_0^{*-}(\zeta^*) + \mathbf{f}_2^+(\zeta^*) + \mathbf{f}_2^-(\zeta^*)]\\
&\quad + \overline{\mathbf{B}}_2[\overline{\mathbf{f}_s^*(\zeta^*)} + \overline{\mathbf{f}_0^{*+}(\zeta^*)} + \overline{\mathbf{f}_0^{*-}(\zeta^*)} + \overline{\mathbf{f}_2^+(\zeta^*)} + \overline{\mathbf{f}_2^-(\zeta^*)}],
\end{aligned} \tag{6.4}$$

Again, as described in the statement following (5.7), to determine the unknown function vectors $\mathbf{f}_s(\zeta)$, $\mathbf{f}_0^+(\zeta)$, $\mathbf{f}_0^-(\zeta)$, $\mathbf{f}_1(\zeta)$, $\mathbf{f}_2^+(\zeta^*)$ and $\mathbf{f}_2^-(\zeta^*)$, only the displacement and traction continuity conditions will be considered because $\mathbf{f}_0^*(\zeta^*) = \mathbf{f}_0^{*+}(\zeta^*) + \mathbf{f}_0^{*-}(\zeta^*) + \mathbf{f}_s^*(\zeta^*)$ shown in (6.1) and (6.2) have satisfied the boundary conditions for the concentrated forces and moments. We now substitute (6.4) into (3.8) with $\zeta_\alpha = \sigma$ along the interface, and separate its results into three parts: (1) the terms related to $\mathbf{f}_s^*(\zeta^*)$ which has singular points in both regions S_1 and $S_2 + S_0$, (2) the terms related to $\mathbf{f}_0^{*-}(\zeta)$ which is not holomorphic in S_1, (3) all the others. By comparison of corresponding terms we obtain

$$\begin{aligned}
&\mathbf{A}_1\mathbf{f}_s(\sigma) + \overline{\mathbf{A}}_1\overline{\mathbf{f}_s(\sigma)} = \mathbf{A}_2\mathbf{f}_s^*(\sigma) + \overline{\mathbf{A}}_2\overline{\mathbf{f}_s^*(\sigma)},\\
&\mathbf{B}_1\mathbf{f}_s(\sigma) + \overline{\mathbf{B}}_1\overline{\mathbf{f}_s(\sigma)} = \mathbf{B}_2\mathbf{f}_s^*(\sigma) + \overline{\mathbf{B}}_2\overline{\mathbf{f}_s^*(\sigma)},
\end{aligned} \tag{6.5a}$$

$$\begin{aligned}
&\mathbf{A}_1\mathbf{f}_0^+(\sigma) + \mathbf{A}_1\mathbf{f}_0^-(\sigma) + \overline{\mathbf{A}}_1\overline{\mathbf{f}_0^+(\sigma)} + \overline{\mathbf{A}}_1\overline{\mathbf{f}_0^-(\sigma)} = \mathbf{A}_2\mathbf{f}_0^{*-}(\sigma) + \overline{\mathbf{A}}_2\overline{\mathbf{f}_0^{*-}(\sigma)},\\
&\mathbf{B}_1\mathbf{f}_0^+(\sigma) + \mathbf{B}_1\mathbf{f}_0^-(\sigma) + \overline{\mathbf{B}}_1\overline{\mathbf{f}_0^+(\sigma)} + \overline{\mathbf{B}}_1\overline{\mathbf{f}_0^-(\sigma)} = \mathbf{B}_2\mathbf{f}_0^{*-}(\sigma) + \overline{\mathbf{B}}_2\overline{\mathbf{f}_0^{*-}(\sigma)},
\end{aligned} \tag{6.5b}$$

and

$$\begin{aligned}&\mathbf{A}_1\mathbf{f}_1(\sigma)-\mathbf{A}_2[\mathbf{f}_0^{*+}(\sigma)+\mathbf{f}_2^{+}(\sigma)]-\overline{\mathbf{A}}_2\overline{\mathbf{f}_2^{-}(\sigma)}\\&\quad=-\overline{\mathbf{A}}_1\overline{\mathbf{f}_1(\sigma)}+\overline{\mathbf{A}}_2[\overline{\mathbf{f}_0^{*+}(\sigma)}+\overline{\mathbf{f}_2^{+}(\sigma)}]+\mathbf{A}_2\mathbf{f}_2^{-}(\sigma),\\&\mathbf{B}_1\mathbf{f}_1(\sigma)-\mathbf{B}_2[\mathbf{f}_0^{*+}(\sigma)+\mathbf{f}_2^{+}(\sigma)]-\overline{\mathbf{B}}_2\overline{\mathbf{f}_2^{-}(\sigma)}\\&\quad=-\overline{\mathbf{B}}_1\overline{\mathbf{f}_1(\sigma)}+\overline{\mathbf{B}}_2[\overline{\mathbf{f}_0^{*+}(\sigma)}+\overline{\mathbf{f}_2^{+}(\sigma)}]+\mathbf{B}_2\mathbf{f}_2^{-}(\sigma).\end{aligned}\tag{6.5c}$$

With $\mathbf{f}_s^*(\zeta^*)$ and $\mathbf{f}_0^{*-}(\zeta^*)$ given in (6.2) and $\mathbf{f}_s(\zeta)$, $\mathbf{f}_0^{+}(\zeta)$ and $\mathbf{f}_0^{-}(\zeta)$ assumed in (6.3), Eqs. (6.5a) and (6.5b) can now provide us the solutions of the unknown coefficients $\mathbf{d}$, $\mathbf{d}_0$, $\mathbf{d}_{-1}$, $\mathbf{d}_1$, $\mathbf{k}_{-1}$ and $\mathbf{k}_1$of functions $\mathbf{f}_s(\zeta)$, $\mathbf{f}_0^{+}(\zeta)$ and $\mathbf{f}_0^{-}(\zeta)$. They are (please refer to Appendix A for detailed derivation)

$$\text{Case 1. } \mathbf{d}=\frac{1}{2\pi\mathrm{i}}\mathbf{A}_1^{\mathrm{T}}\hat{\mathbf{p}};\tag{6.6a}$$

$$\begin{aligned}\text{Case 2. } \mathbf{d}_0&=\frac{-\hat{f}_3}{\pi\mathrm{i}}\{\mathbf{B}_1^{\mathrm{T}}\mathbf{J}_{2R}+\mathbf{A}_1^{\mathrm{T}}\mathbf{J}_{1R}^{\mathrm{T}}\}\mathbf{i}_3,\\ \mathbf{d}_1&=\{(\mathbf{B}_1^{\mathrm{T}}\mathbf{g}_2+\mathbf{A}_1^{\mathrm{T}}\mathbf{h}_2)+\mathrm{i}(\mathbf{B}_1^{\mathrm{T}}\mathbf{g}_1+\mathbf{A}_1^{\mathrm{T}}\mathbf{h}_1)\},\\ \mathbf{d}_{-1}&=\{-(\mathbf{B}_1^{\mathrm{T}}\mathbf{g}_2+\mathbf{A}_1^{\mathrm{T}}\mathbf{h}_2)+\mathrm{i}(\mathbf{B}_1^{\mathrm{T}}\mathbf{g}_1+\mathbf{A}_1^{\mathrm{T}}\mathbf{h}_1)\},\\ \mathbf{k}_1&=\frac{\hat{f}_3}{2\pi\mathrm{i}}\{(\mathbf{B}_1^{\mathrm{T}}\mathbf{Q}_{2R}+\mathbf{A}_1^{\mathrm{T}}\mathbf{Q}_{1R}^{\mathrm{T}})+\mathrm{i}(\mathbf{B}_1^{\mathrm{T}}\mathbf{Q}_{2I}+\mathbf{A}_1^{\mathrm{T}}\mathbf{Q}_{1I}^{\mathrm{T}})\}\mathbf{i}_3,\\ \mathbf{k}_{-1}&=\frac{\hat{f}_3}{2\pi\mathrm{i}}\{-(\mathbf{B}_1^{\mathrm{T}}\mathbf{Q}_{2R}+\mathbf{A}_1^{\mathrm{T}}\mathbf{Q}_{1R}^{\mathrm{T}})+\mathrm{i}(\mathbf{B}_1^{\mathrm{T}}\mathbf{Q}_{2I}+\mathbf{A}_1^{\mathrm{T}}\mathbf{Q}_{1I}^{\mathrm{T}})\}\mathbf{i}_3.\end{aligned}\tag{6.6b}$$

$$\text{Case 3. None.}\tag{6.6c}$$

In the above,

$$\begin{Bmatrix}\mathbf{g}_1\\ \mathbf{h}_1\end{Bmatrix}=\frac{-\hat{f}_3}{2\pi}\begin{Bmatrix}(\mathbf{E}_{2R}+\mathbf{F}_{2R})\mathbf{i}_3\\ (\mathbf{E}_{1R}^{\mathrm{T}}+\mathbf{F}_{1R}^{\mathrm{T}})\mathbf{i}_3\end{Bmatrix},\quad \begin{Bmatrix}\mathbf{g}_2\\ \mathbf{h}_2\end{Bmatrix}=\frac{\hat{f}_3}{2\pi}\begin{Bmatrix}(\mathbf{E}_{2I}-\mathbf{F}_{2I})\mathbf{i}_3\\ (\mathbf{E}_{1I}^{\mathrm{T}}-\mathbf{F}_{1I}^{\mathrm{T}})\mathbf{i}_3\end{Bmatrix},\tag{6.7}$$

where $\mathbf{E}_{kR}$, $\mathbf{F}_{kR}$, $\mathbf{J}_{kR}$, $\mathbf{Q}_{kR}$ and $\mathbf{E}_{kI}$, $\mathbf{F}_{kI}$, $\mathbf{J}_{kI}$, $\mathbf{Q}_{kI}$, k=1,2,3, are real and imaginary parts of the matrices $\mathbf{E}_k$, $\mathbf{F}_k$, $\mathbf{J}_k$ and $\mathbf{Q}_k$, which is defined as

$$\begin{bmatrix}\mathbf{E}_1&\mathbf{E}_2\\ \mathbf{E}_3&\mathbf{E}_1^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_2<c_\alpha^*>\mathbf{B}_2^{\mathrm{T}}&\mathbf{A}_2<c_\alpha^*>\mathbf{A}_2^{\mathrm{T}}\\ \mathbf{B}_2<c_\alpha^*>\mathbf{B}_2^{\mathrm{T}}&\mathbf{B}_2<c_\alpha^*>\mathbf{A}_2^{\mathrm{T}}\end{bmatrix},\tag{6.8a}$$

$$\begin{bmatrix}\mathbf{F}_1&\mathbf{F}_2\\ \mathbf{F}_3&\mathbf{F}_1^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_2<c_\alpha^*\gamma_\alpha^*>\mathbf{B}_2^{\mathrm{T}}&\mathbf{A}_2<c_\alpha^*\gamma_\alpha^*>\mathbf{A}_2^{\mathrm{T}}\\ \mathbf{B}_2<c_\alpha^*\gamma_\alpha^*>\mathbf{B}_2^{\mathrm{T}}&\mathbf{B}_2<c_\alpha^*\gamma_\alpha^*>\mathbf{A}_2^{\mathrm{T}}\end{bmatrix},\tag{6.8b}$$

$$\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\ \mathbf{J}_3&\mathbf{J}_1^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_2<c_\alpha^*(\hat{\zeta}_\alpha^*+\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*})>\mathbf{B}_2^{\mathrm{T}}&\mathbf{A}_2<c_\alpha^*(\hat{\zeta}_\alpha^*+\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*})>\mathbf{A}_2^{\mathrm{T}}\\ \mathbf{B}_2<c_\alpha^*(\hat{\zeta}_\alpha^*+\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*})>\mathbf{B}_2^{\mathrm{T}}&\mathbf{B}_2<c_\alpha^*(\hat{\zeta}_\alpha^*+\frac{\gamma_\alpha^*}{\hat{\zeta}_\alpha^*})>\mathbf{A}_2^{\mathrm{T}}\end{bmatrix},\tag{6.8c}$$

$$\begin{bmatrix}\mathbf{Q}_1&\mathbf{Q}_2\\ \mathbf{Q}_3&\mathbf{Q}_1^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_2<c_\alpha^*(\log c_\alpha^*-1)>\mathbf{B}_2^{\mathrm{T}}&\mathbf{A}_2<c_\alpha^*(\log c_\alpha^*-1)>\mathbf{A}_2^{\mathrm{T}}\\ \mathbf{B}_2<c_\alpha^*(\log c_\alpha^*-1)>\mathbf{B}_2^{\mathrm{T}}&\mathbf{B}_2<c_\alpha^*(\log c_\alpha^*-1)>\mathbf{A}_2^{\mathrm{T}}\end{bmatrix},\tag{6.8d}$$

$$\begin{aligned}&\mathbf{E}_k=\mathbf{E}_{kR}+\mathrm{i}\mathbf{E}_{kI},\quad \mathbf{F}_k=\mathbf{F}_{kR}+\mathrm{i}\mathbf{F}_{kI},\\&\mathbf{J}_k=\mathbf{J}_{kR}+\mathrm{i}\mathbf{J}_{kI},\quad \mathbf{Q}_k=\mathbf{Q}_{kR}+\mathrm{i}\mathbf{Q}_{kI},\quad i=1,2,3.\end{aligned}\tag{6.8e}$$

From (6.5c) and following the standard approach of analytical continuation as that shown in Section 5 and (Hwu and Yen, 1993; Hwu, 2005), the unknown function vectors $\mathbf{f}_1(\zeta)$ and $\mathbf{f}_2(\zeta^*)$ can be determined as

$$\mathbf{f}_2(\zeta^*)=\sum_{k=-\infty}^{\infty}\mathbf{c}_k\zeta^{*k},\mathbf{c}_{-k}=<\gamma_\alpha^{*k}>\mathbf{c}_k,\tag{6.9}$$

and

$$\mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} \mathbf{A}_1^{-1}\{\mathbf{A}_2\mathbf{e}_k^+ + \mathbf{A}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k + \overline{\mathbf{A}}_2\overline{\mathbf{c}}_k\}\zeta^{-k}, \tag{6.10a}$$

or

$$\mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} \mathbf{B}_1^{-1}\{\mathbf{B}_2\mathbf{e}_k^+ + \mathbf{B}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k + \overline{\mathbf{B}}_2\overline{\mathbf{c}}_k\}\zeta^{-k}, \tag{6.10b}$$

where

$$\mathbf{c}_k = \{\mathbf{G}_0 - \overline{\mathbf{G}}_k\overline{\mathbf{G}}_0^{-1}\mathbf{G}_k\}^{-1}\{\mathbf{t}_k - \overline{\mathbf{G}}_k\overline{\mathbf{G}}_0^{-1}\bar{\mathbf{t}}_k\}, \quad k = 1, 2, \ldots, \infty, \tag{6.11a}$$

and

$$\mathbf{t}_k = -(\overline{\mathbf{M}}_1 - \overline{\mathbf{M}}_2)\overline{\mathbf{A}}_2\bar{\mathbf{e}}_k^+. \tag{6.11b}$$

In the above, $\mathbf{e}_k^+$ are the coefficients of the Taylor's expansions of $\mathbf{f}_0^{*+}(\zeta^*)$, i.e.,

$$\mathbf{f}_0^{*+}(\zeta) = \sum_{k=1}^{\infty} \mathbf{e}_k^+\zeta^{-k}. \tag{6.12}$$

From Eqs. ((6.2a)–(6.2c)), we have

Case 1.

$$\mathbf{e}_k^+ = \frac{-1}{2\pi k i} < \hat{\zeta}_\alpha^{*k} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^k > \mathbf{A}_2^{\mathrm{T}}\hat{\mathbf{p}}, \tag{6.13a}$$

Case 2.

$$\begin{aligned} \mathbf{e}_1^+ &= \frac{\hat{f}_3}{2\pi \mathrm{i}} < c_\alpha^*\left\{\frac{1}{2}[\hat{\zeta}_\alpha^{*2} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^2] + \gamma_\alpha^*(\log c_\alpha^* + 1)\right\} > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3, \\ \mathbf{e}_k^+ &= \frac{\hat{f}_3}{2\pi \mathrm{i}} < \frac{c_\alpha^*}{k}\left\{\frac{1}{k+1}[\hat{\zeta}_\alpha^{*k+1} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^{k+1}] - \frac{\gamma_\alpha^*}{k-1}[\hat{\zeta}_\alpha^{*k-1} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^{k-1}]\right\} > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3, \quad k \neq 1, \end{aligned} \tag{6.13b}$$

Case 3.

$$\mathbf{e}_k^+ = \frac{\hat{m}_3}{2\pi \mathrm{i}} < \frac{\hat{\zeta}_\alpha^{*k} - (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^k}{c_\alpha^*[\hat{\zeta}_\alpha^* - (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)]} > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_2, \tag{6.13c}$$

Same as Section 5, the complete full field solutions for the Green's functions with forces/moments inside the inclusion are compiled and shown in Table 3. Again, the solutions obtained here for Case 1 possess exactly the same mathematical form as those for the corresponding two-dimensional problems. The only difference is the contents of the symbols.

Following the steps of Section 5, the stress resultants and moments along the interface boundary can also be obtained through the explicit solution of $\boldsymbol{\phi}_{d,s}$, which is

$$\boldsymbol{\phi}_{d,s} = \boldsymbol{\phi}_{d,s}^{(1)} = \boldsymbol{\phi}_{d,s}^{(2)} = 2\mathrm{Re}\left\{\mathbf{B}_2\mathbf{f}_{0,s}^* + \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}k\omega}}{\rho}\mathbf{B}_2\mathbf{c}_k\right\}, \quad \text{along the interface}, \tag{6.14a}$$

where

$$\mathbf{f}_{0,s}^* = \begin{cases} \frac{1}{2\pi \mathrm{i}} < (\cos\theta + \mu_\alpha^*\sin\theta)\ell_\alpha^{-1} > \mathbf{A}_2^{\mathrm{T}}\hat{\mathbf{p}}, & \text{case 1}, \\ \frac{\hat{f}_3}{2\pi \mathrm{i}} < (\cos\theta + \mu_\alpha^*\sin\theta)\log\ell_\alpha > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_3, & \text{case 2}, \\ \frac{-\hat{m}_3}{2\pi \mathrm{i}} < (\cos\theta + \mu_\alpha^*\sin\theta)\ell_\alpha^{-2} > \mathbf{A}_2^{\mathrm{T}}\mathbf{i}_2, & \text{case 3}. \end{cases} \tag{6.14b}$$

and

$$\ell_\alpha = a\cos\omega + \mu_\alpha^* b\sin\omega - \hat{z}_\alpha^*. \tag{6.14c}$$

Table 3
Green's functions for inclusion problems—loads inside the inclusions

Two-dimensional problems: coupled inplane–antiplane deformations (Yen et al., 1995) Point load: $\hat{f}_1, \hat{f}_2, \hat{f}_3$

$$\mathbf{u}_1 = 2\mathrm{Re}\{\mathbf{A}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, \quad \boldsymbol{\phi}_1 = 2\mathrm{Re}\{\mathbf{B}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\},$$
$$\mathbf{u}_2 = 2\mathrm{Re}\{\mathbf{A}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\}, \quad \boldsymbol{\phi}_2 = 2\mathrm{Re}\{\mathbf{B}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\},$$
$$\mathbf{u}_1, \mathbf{u}_2 : 3\times 1, \quad \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 : 3\times 1, \mathbf{A}_1, \mathbf{A}_2 : 3\times 3, \quad \mathbf{B}_1, \mathbf{B}_2 : 3\times 3; \quad \mathbf{f}_0(\zeta), \mathbf{f}_1(\zeta), \mathbf{f}_0^*(\zeta^*), \mathbf{f}_2(\zeta^*) : 3\times 1$$

$$\mathbf{f}_0(\zeta) = \frac{1}{2\pi i} < \log \zeta_\alpha > \mathbf{A}_1^{\mathrm{T}} \hat{\mathbf{p}}$$
$$\mathbf{f}_0^*(\zeta^*) = \frac{1}{2\pi i} < \log(z_\alpha^* - \hat{z}_\alpha^*) > \mathbf{A}_2^{\mathrm{T}} \hat{\mathbf{p}}$$
$$\mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{A}_1^{-1}\{\mathbf{A}_2\mathbf{e}_k^+ + \mathbf{A}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k + \overline{\mathbf{A}}_2\overline{\mathbf{c}}_k\}, \text{ or } \mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{B}_1^{-1}\{\mathbf{B}_2\mathbf{e}_k^+ + \mathbf{B}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k + \overline{\mathbf{B}}_2\overline{\mathbf{c}}_k\}$$
$$\mathbf{f}_2(\zeta^*) = \sum_{k=-\infty}^{\infty} < \zeta_\alpha^{*k} > \mathbf{c}_k, \mathbf{c}_{-k} = < \gamma_\alpha^{*k} > \mathbf{c}_k$$

in which

$$\mathbf{e}_k^+ = \frac{-1}{2\pi k i} < \hat{\zeta}_\alpha^{*k} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^k > \mathbf{A}_2^{\mathrm{T}} \hat{\mathbf{p}},$$

Coupled stretching–bending problems Case 1: $\hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2$; Case 2: $\hat{f}_3$; Case 3: $\hat{m}_3$

$$\mathbf{u}_d^{(1)} = 2\mathrm{Re}\{\mathbf{A}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\}, \boldsymbol{\phi}_d^{(1)} = 2\mathrm{Re}\{\mathbf{B}_1[\mathbf{f}_0(\zeta) + \mathbf{f}_1(\zeta)]\},$$
$$\mathbf{u}_d^{(2)} = 2\mathrm{Re}\{\mathbf{A}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\}, \boldsymbol{\phi}_d^{(2)} = 2\mathrm{Re}\{\mathbf{B}_2[\mathbf{f}_0^*(\zeta^*) + \mathbf{f}_2(\zeta^*)]\},$$
$$\mathbf{u}_d^{(1)}, \mathbf{u}_d^{(2)} : 4\times 1, \quad \boldsymbol{\phi}_d^{(1)}, \boldsymbol{\phi}_d^{(2)} : 4\times 1, \mathbf{A}_1, \mathbf{A}_2 : 4\times 4, \quad \mathbf{B}_1, \mathbf{B}_2 : 4\times 4; \quad \mathbf{f}_0(\zeta), \mathbf{f}_1(\zeta), \mathbf{f}_0^*(\zeta^*), \mathbf{f}_2(\zeta^*) : 4\times 1$$

Case 1: $\mathbf{f}_0(\zeta) = \frac{1}{2\pi i} < \log \zeta_\alpha > \mathbf{A}_1^{\mathrm{T}} \hat{\mathbf{p}}$

$$\mathbf{f}_0^*(\zeta^*) = \frac{1}{2\pi i} < \log(z_\alpha^* - \hat{z}_\alpha^*) > \mathbf{A}_2^{\mathrm{T}} \hat{\mathbf{p}}$$
$$\mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{A}_1^{-1}\{\mathbf{A}_2\mathbf{e}_k^+ + \mathbf{A}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k + \overline{\mathbf{A}}_2\overline{\mathbf{c}}_k\}, \text{ or } \mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} < \zeta_\alpha^{-k} > \mathbf{B}_1^{-1}\{\mathbf{B}_2\mathbf{e}_k^+ + \mathbf{B}_2 < \gamma_\alpha^{*k} > \mathbf{c}_k + \overline{\mathbf{B}}_2\overline{\mathbf{c}}_k\}$$
$$\mathbf{f}_2(\zeta^*) = \sum_{k=-\infty}^{\infty} < \zeta_\alpha^{*k} > \mathbf{c}_k, \mathbf{c}_{-k} = < \gamma_\alpha^{*k} > \mathbf{c}_k$$

Case 2: $\mathbf{f}_0(\zeta) = < \log \zeta_\alpha > \{< \zeta_\alpha > \mathbf{d}_1 + < \zeta_\alpha^{-1} > \mathbf{d}_{-1} + \mathbf{d}_0\} + < \zeta_\alpha^{-1} > \mathbf{k}_{-1} + < \zeta_\alpha > \mathbf{k}_1$

$$\mathbf{f}_0^*(\zeta^*) = \frac{\hat{f}_3}{2\pi i} < (z_\alpha^* - \hat{z}_\alpha^*)[\log(z_\alpha^* - \hat{z}_\alpha^*) - 1] > \mathbf{A}_2^{\mathrm{T}} \mathbf{i}_3$$

$\mathbf{f}_1(\zeta)$, $\mathbf{f}_2(\zeta^*)$: same mathematical form as Case 1, the only difference is the content of $\mathbf{e}_k^+$ and $\mathbf{c}_k$

$\mathbf{d}_1, \mathbf{d}_{-1}, \mathbf{d}_0, \mathbf{k}_{-1}, \mathbf{k}_1$: Eqs.(6.6b), (6.7), (6.8a–e)

Case 3: $\mathbf{f}_0(\zeta) = \mathbf{0}$

$$\mathbf{f}_0^*(\zeta^*) = \frac{\hat{m}_3}{2\pi i} < \frac{1}{z_\alpha^* - \hat{z}_\alpha^*} > \mathbf{A}_2^{\mathrm{T}} \mathbf{i}_2$$

$\mathbf{f}_1(\zeta)$, $\mathbf{f}_2(\zeta^*)$: same mathematical form as Case 1, the only difference is the content of $\mathbf{e}_k^+$ and $\mathbf{c}_k$

in which

Case 1: $\mathbf{e}_k^+ = \frac{-1}{2\pi k i} < \hat{\zeta}_\alpha^{*k} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^k > \mathbf{A}_2^{\mathrm{T}} \hat{\mathbf{p}},$

Case 2: $\mathbf{e}_1^+ = \frac{\hat{f}_3}{2\pi i} < c_\alpha^* \left\{ \frac{1}{2} \left[\hat{\zeta}_\alpha^{*2} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^2 \right] + \gamma_\alpha^* (\log c_\alpha^* + 1) \right\} > \mathbf{A}_2^{\mathrm{T}} \mathbf{i}_3,$

$$\mathbf{e}_k^+ = \frac{\hat{f}_3}{2\pi i} < \frac{c_\alpha^*}{k} \left\{ \frac{1}{k+1} \left[\hat{\zeta}_\alpha^{*k+1} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^{k+1} \right] - \frac{\gamma_\alpha^*}{k-1} \left[\hat{\zeta}_\alpha^{*k-1} + (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^{k-1} \right] \right\} > \mathbf{A}_2^{\mathrm{T}} \mathbf{i}_3, k \neq 1,$$

Case 3: $\mathbf{e}_k^+ = \frac{\hat{m}_3}{2\pi i} < \frac{\hat{\zeta}_\alpha^{*k} - (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)^k}{c_\alpha^*[\hat{\zeta}_\alpha^* - (\gamma_\alpha^*/\hat{\zeta}_\alpha^*)]} > \mathbf{A}_2^{\mathrm{T}} \mathbf{i}_2,$

$\mathbf{c}_k$: same as that shown in Table 2 except $\mathbf{t}_k = -(\overline{\mathbf{M}}_1 - \overline{\mathbf{M}}_2)\overline{\mathbf{A}}_2\overline{\mathbf{e}}_k^+$. Refer to Tables 2 and 4 for the other symbols.

7. Verification and discussions

From the statements and the solutions given in the previous sections, we know that Green's function is a solution to the incomplete problem whose domain is infinite and whose loading is an unbalanced point load. Based upon this solution, we may construct the solutions for any complete problem whose boundary is prescribed by tractions or displacements and all applied forces and reactive forces are in equilibrium condition. In other words, the complete problem may be solved through superposition of several Green's function with different intensities located at different points. That is why the Green's function is sometimes called the fundamental solution for the boundary element method. Therefore, in this stage it is not appropriate to compare our solutions with those obtained from the other numerical methods such as the finite element method since they can only deal with complete problems. For two-dimensional problems, a special boundary element for holes/cracks/inclusions has been developed based upon the associated Green's function, and several numerical examples have also been done and compared (Hwu and Liao, 1994). Following this concept, a special boundary element for the coupled stretching–bending analysis of holes/cracks/inclusions will be developed (Hwu and Liang, 2006). Like the two-dimensional problems, the efficiency and accuracy of this special boundary element over the general finite element can then be expected.

Due to the inappropriateness of numerical check by finite element method, in this paper we first consider the analytical verification with some special cases reduced from our general solutions, and then present the numerical results for various elastic inclusions. Since no analytical solution has been presented in the literature for the general cases discussed in this paper, to check the correctness of our derivation we consider: (1) the simplest condition that the matrix and the inclusion are composed of the same materials, i.e., no inclusions are embedded in the laminates; (2) the case that the inclusion is very soft, which can be checked by the results of corresponding hole problems. The analytical solutions of these two special cases can be found in (Hwu, 2004) and (Hwu, 2005).

7.1. Without inclusions

7.1.1. Specialization of the solutions from Section 5

If the matrix and the inclusion are composed of the same material, the material eigenvector matrices should be identical, and hence we can set $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}$ and $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$. Thus, from (5.13c) we have $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}$. From the identities shown in (Ting, 1996), we know that the impedance matrix $\mathbf{M}$ is related to the Stroh fundamental matrices $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{L}$, which are real matrices, by

$$\mathbf{M} = \mathbf{H}^{-1}(\mathbf{I} + \mathrm{i}\mathbf{S}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}). \tag{7.1}$$

By using the above relations, the coefficient vector $\mathbf{c}_k$ and the function vector $\breve{\mathbf{f}}_1(\zeta)$ obtained in (5.13) and (5.14) can now be reduced to

$$\mathbf{c}_k = \mathbf{e}_k^-, \ \breve{\mathbf{f}}_1(\zeta) = \sum_{k=1}^{\infty} < (\gamma_\alpha/\zeta_\alpha)^k > \mathbf{e}_k^-. \tag{7.2}$$

Note that in deriving the second equation of (7.2), the translating technique stated in the paragraph following (5.8) has been employed. The coefficient vector $\mathbf{e}_k^-$ for different loading cases has been given in ((5.10a)–(5.10c)). From $(7.2)_1$, (5.9) and $(5.11)_2$, we see that $\mathbf{f}_2^-(\zeta) = \mathbf{f}_0^-(\zeta)$ whose solutions for three different loading cases have been shown in (5.3a)–(5.3c). With the results of 5.10a,5.10b,5.10c and (7.2), the function vectors $\breve{\mathbf{f}}_1(\zeta)$ and $\mathbf{f}_2^+(\zeta)$ can be obtained from $(7.2)_2$ and $(5.11)_1$. Their final simplified results are

Case 1.

$$\breve{\mathbf{f}}_1(\zeta) = \mathbf{f}_2^+(\zeta) = \frac{1}{2\pi \mathrm{i}} < \log\left(1 - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha}\right) > \mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}}. \tag{7.3a}$$

Case 2.

$$\breve{\mathbf{f}}_1(\zeta) = \mathbf{f}_2^+(\zeta) = \frac{\hat{f}_3}{2\pi i} < \frac{c_\alpha \gamma_\alpha}{\zeta_\alpha}\left\{\log(-c_\alpha \hat{\zeta}_\alpha) + \frac{\zeta_\alpha - \hat{\zeta}_\alpha}{\hat{\zeta}_\alpha}\right\} + (z_\alpha - \hat{z}_\alpha)\log\left(1 - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha}\right) > \mathbf{A}^T \mathbf{i}_3. \tag{7.3b}$$

Case 3.

$$\breve{\mathbf{f}}_1(\zeta) = \mathbf{f}_2^+(\zeta) = \frac{-\hat{m}_3}{2\pi i} < \frac{c_{4\alpha}(\gamma_\alpha / \hat{\zeta}_\alpha \zeta_\alpha)}{1 - (\gamma_\alpha / \hat{\zeta}_\alpha \zeta_\alpha)} > \mathbf{A}^T \mathbf{i}_2. \tag{7.3c}$$

Note that in deriving the solutions of (7.3), the following relations have been used

$$z_\alpha - \hat{z}_\alpha = c_\alpha\{\zeta_\alpha - \hat{\zeta}_\alpha + \gamma_\alpha(\zeta_\alpha^{-1} - \hat{\zeta}_\alpha^{-1})\} = c_\alpha(\zeta_\alpha - \hat{\zeta}_\alpha)\left(1 - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha}\right),$$
$$\log\left(1 - \frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha}\right) = -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha}\right)^k, \quad \text{for} \left|\frac{\gamma_\alpha}{\hat{\zeta}_\alpha \zeta_\alpha}\right| < 1. \tag{7.4}$$

Combining the results of (5.3) and (7.3), it can be shown that

$$\mathbf{f}_0^-(\zeta) + \breve{\mathbf{f}}_1(\zeta) = \mathbf{f}_2^+(\zeta) + \mathbf{f}_2^-(\zeta) = \mathbf{f}_0(z), \tag{7.5}$$

in which $\mathbf{f}_0(z)$ is given in (5.2a)–(5.2c) for three different loading cases. In other words, our solutions have been successfully reduced to the cases of homogeneous laminates.

7.1.2. Specialization of the solutions from Section 6

Similar to the reduction process stated above, by setting $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}$ and $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$ the solutions obtained in (6.10) and (6.11) may now be reduced to

$$\mathbf{c}_k = \mathbf{0}, \quad \mathbf{f}_1(\zeta) = \sum_{k=1}^{\infty} \mathbf{e}_k^+ \zeta^{-k} = \mathbf{f}_0^{*+}(\zeta). \tag{7.6}$$

Combining the results of (7.6), (6.2), (6.3) and (6.6), and using the relations given in (A.3), we can prove that

$$\mathbf{f}_s(\zeta) + \mathbf{f}_0^+(\zeta) + \mathbf{f}_0^-(\zeta) + \mathbf{f}_1(\zeta) = \mathbf{f}_0(z), \tag{7.7}$$

in which $\mathbf{f}_0(z)$ is given in (5.2a)–(5.2c) or (6.1a)–(6.1c) for three different loading cases. To prove that the other part of the general solution (6.4) also reduces to the solution of homogeneous laminates, we combine the results of $(7.6)_1$, (6.9) and (6.2), which also give us

$$\mathbf{f}_s^*(\zeta^*) + \mathbf{f}_0^{*+}(\zeta^*) + \mathbf{f}_0^{*-}(\zeta^*) + \mathbf{f}_2^+(\zeta^*) + \mathbf{f}_2^-(\zeta^*) = \mathbf{f}_0(z). \tag{7.8}$$

7.2. Holes

If the inclusion is replaced by a hole, the elastic constants C_{ijkl} of the inclusion can be set to be zero, and then the extensional, coupling and bending stiffness $A_{ijkl}, B_{ijkl}, D_{ijkl}$ will all be equal to zero. Substituting these zero values into the explicit solutions of the material eigenvector matrices $\mathbf{A}$ and $\mathbf{B}$ (Hwu, 2003a,b), we get $\mathbf{B}_2 = \mathbf{0}$ whereas $\mathbf{A}_2$ may not be a zero matrix. With $\mathbf{B}_2 = \mathbf{0}$, the solution $\breve{\mathbf{f}}_1(\zeta)$ obtained in (5.14b) can be reduced to

$$\breve{\mathbf{f}}_1(\zeta) = -\sum_{k=1}^{\infty} \mathbf{B}_1^{-1}\overline{\mathbf{B}}_1 \bar{\mathbf{e}}_k^- \zeta^{-k}. \tag{7.9}$$

Substituting $\mathbf{e}_k^-$ of (5.10a)–(5.10c) into (7.9), using the following Taylor series relations, and employing the translating technique (Hwu, 1993),

$$-\sum_{k=1}^{\infty} \frac{x^k}{k} = \log(1 - x), \quad \sum_{k=1}^{\infty} x^k = \frac{1}{1 - x}, \quad \text{for} |x| < 1, \tag{7.10}$$

Table 4
Green's functions for hole/crack problems

Two-dimensional problems: coupled inplane–antiplane deformations	Point load: $\hat{f}_1, \hat{f}_2, \hat{f}_3$
$\mathbf{u} = 2\mathrm{Re}\{\mathbf{A}\mathbf{f}(z)\}$, $\boldsymbol{\phi} = 2\mathrm{Re}\{\mathbf{B}\mathbf{f}(z)\}$,	$\mathbf{u}{:}3\times 1$, $\boldsymbol{\phi}{:}3\times 1$, $\mathbf{A}{:}3\times 3$, $\mathbf{B}{:}3\times 3$; $\mathbf{f}(z){:}3\times 1$
Without holes (Ting, 1996)	$\mathbf{f}(z) = \frac{1}{2\pi i} < \log(z_\alpha - \hat{z}_\alpha) > \mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}}, \quad \hat{\mathbf{p}} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)^{\mathrm{T}}$
Holes/cracks (Hwu and Yen, 1991)	$\mathbf{f}(\zeta) = \frac{1}{2\pi i}\left\{ < \log(\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{A}^{\mathrm{T}} + \sum_{k=1}^{3} < \log(\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k) > \mathbf{B}^{-1}\overline{\mathbf{B}}I_k\overline{\mathbf{A}}^{\mathrm{T}} \right\}\hat{\mathbf{p}}$
Coupled stretching–bending problems $\mathbf{u}_d = 2\mathrm{Re}\{\mathbf{A}\mathbf{f}(z)\}$, $\boldsymbol{\phi}_d = 2\mathrm{Re}\{\mathbf{B}\mathbf{f}(z)\}$ Without Holes (Hwu, 2004)	Case 1: $\hat{f}_1, \hat{f}_2, \hat{m}_1, \hat{m}_2$; Case 2: $\hat{f}_3$; Case 3: $\hat{m}_3$ $\mathbf{u}_d{:}4\times 1$, $\boldsymbol{\phi}_d{:}4\times 1$, $\mathbf{A}{:}4\times 4$, $\mathbf{B}{:}4\times 4$; $\mathbf{f}(z){:}4\times 1$ Case 1: $\mathbf{f}(z) = \frac{1}{2\pi i} < \log(z_\alpha - \hat{z}_\alpha) > \mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}}, \quad \hat{\mathbf{p}} = (\hat{f}_1, \hat{f}_2, \hat{m}_2, -\hat{m}_1)^{\mathrm{T}}$ Case 2: $\mathbf{f}(z) = \frac{\hat{f}_3}{2\pi i} < (z_\alpha - \hat{z}_\alpha)[\log(z_\alpha - \hat{z}_\alpha) - 1] > \mathbf{A}^{\mathrm{T}}\mathbf{i}_3, \quad \mathbf{i}_3 = (0,0,1,0)^{\mathrm{T}}$ Case 3: $\mathbf{f}(z) = \frac{\hat{m}_3}{2\pi i}\frac{1}{z_\alpha - \hat{z}_\alpha}\mathbf{A}^{\mathrm{T}}\mathbf{i}_2, \quad \mathbf{i}_2 = (0,1,0,0)^{\mathrm{T}}$
Holes/Cracks (Hwu, 2005)	Case 1: $\mathbf{f}(\zeta) = \frac{1}{2\pi i}\left\{ < \log(\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{A}^{\mathrm{T}} + \sum_{k=1}^{4} < \log(\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k) > \mathbf{B}^{-1}\overline{\mathbf{B}}I_k\overline{\mathbf{A}}^{\mathrm{T}} \right\}\hat{\mathbf{p}}$ Case 2: $\begin{aligned} \mathbf{f}(\zeta) = &< (z_\alpha - \hat{z}_\alpha)\log(\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{q}_2 \\ &- \sum_{k=1}^{4} < (\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k)(1 - \bar{\gamma}_k\bar{\hat{\zeta}}_k^{-1}\zeta_\alpha)\log(\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k) > \mathbf{B}^{-1}\overline{\mathbf{B}}I_k\bar{\mathbf{q}}_c \\ &+ < (\zeta_\alpha - \hat{\zeta}_\alpha) > \mathbf{q}_c^* - \sum_{k=1}^{4} < (\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k) > \mathbf{B}^{-1}\overline{\mathbf{B}}I_k\bar{\mathbf{q}}_c^* \\ &- < (\zeta_\alpha^{-1} - \hat{\zeta}_\alpha^{-1}) > \mathbf{q}_c^{**} + \sum_{k=1}^{4} < (\zeta_\alpha^{-}\bar{\hat{\zeta}}_k^{-1}) > \mathbf{B}^{-1}\overline{\mathbf{B}}I_k\bar{\mathbf{q}}_c^{**}. \end{aligned}$ Case 3: $\mathbf{f}(\zeta) = \frac{1}{\zeta_\alpha - \hat{\zeta}_\alpha}\mathbf{q}_3^* - \sum_{k=1}^{4}\frac{1}{\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k}\mathbf{B}^{-1}\overline{\mathbf{B}}I_k\bar{\mathbf{q}}_3^*.$ in which $\mathbf{q}_c = < c_\alpha > \mathbf{q}_2, \mathbf{q}_c^* = < c_{2\alpha} > \mathbf{q}_2, \mathbf{q}_c^{**} = < c_{3\alpha} > \mathbf{q}_2,$ $\mathbf{q}_3^* = < c_{4\alpha}\hat{\zeta}_\alpha > \mathbf{q}_3, \mathbf{q}_3^{**} = < c_{4\alpha}\gamma_\alpha/\hat{\zeta}_\alpha > \mathbf{q}_3,$ $\mathbf{q}_1 = \frac{1}{2\pi i}\mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}}, \mathbf{q}_2 = \frac{\hat{f}_3}{2\pi i}\mathbf{A}^{\mathrm{T}}\mathbf{i}_3, \mathbf{q}_3 = \frac{\hat{m}_3}{2\pi i}\mathbf{A}^{\mathrm{T}}\mathbf{i}_2$

$z_k = x_1 + \mu_k x_2, \quad \hat{z}_k = \hat{x}_1 + \mu_k\hat{x}_2, \quad \gamma_\alpha = \frac{a + ib\mu_\alpha}{a - ib\mu_\alpha}, \zeta_k = \frac{z_k + \sqrt{z_k^2 - a^2 - \mu_k^2 b^2}}{a - i\mu_k b}, \quad \hat{\zeta}_k = \frac{\hat{z}_k + \sqrt{\hat{z}_k^2 - a^2 - \mu_k^2 b^2}}{a - i\mu_k b},$

Crack: $b = 0$ $c_\alpha = \frac{1}{2}(a - ib\mu_\alpha), \quad c_{2\alpha} = c_\alpha(\log c_\alpha - 1), \quad c_{3\alpha} = c_\alpha\gamma_\alpha\log(-\hat{\zeta}_\alpha), \quad c_{4\alpha} = [c_\alpha(\hat{\zeta}_\alpha - \gamma_\alpha/\hat{\zeta}_\alpha)]^{-1},$

$\mathbf{I}_1 = \mathrm{diag}[1,0,0,0], \quad \mathbf{I}_2 = \mathrm{diag}[0,1,0,0], \quad \mathbf{I}_3 = \mathrm{diag}[0,0,1,0], \quad \mathbf{I}_4 = \mathrm{diag}[0,0,\ 0,\ 1].$

we can prove that the solution (7.9) is identical to those shown in Table 4 for all three loading cases of the coupled stretching–bending hole problems.

7.3. Elastic inclusions

After the analytical check through the above two special cases, we now perform the numerical calculation for the cases of general elastic inclusions to see whether the elastic responses shown by our Green's function are conformable to our engineering intuition. Although our solutions are valid for the general composite laminates, symmetric or unsymmetric, to save the space of this paper only the simplest orthotropic lamina is considered in the following examples.

Consider both of the matrix and inclusion are orthotropic materials. The material properties of the matrix are

$$E_1 = 144.8\ \text{GPa}, \quad E_2 = E_3 = 10.7\ \text{GPa}, \quad v_{12} = v_{13} = v_{23} = 0.31,$$
$$G_{12} = G_{13} = G_{23} = 4.5\ \text{GPa}$$

while the properties of the inclusion are assumed to be proportional to the matrix as

$$k = \frac{(E_i)_2}{(E_i)_1} = \frac{(G_{ij})_2}{(G_{ij})_1}, \quad i,j = 1,2,3, \quad v_{12} = v_{13} = v_{23} = 0.31,$$

where k is the index of softness (or hardness). When $k < 1$ the inclusion is softer than the matrix, while for $k > 1$ means hard. A hole or rigid inclusion can therefore be approximated by letting $k \to 0$ or $k \to \infty$. Consider a concentrated force $\hat{p}$ (Nt) directed in the x_2-axis applied on the point $(\hat{x}_1/a, \hat{x}_2/a) = (0, 5)$. The elliptical inclusion is represented by b/a=0.75. The hoop stress resultant N_s of elastic inclusion for various k are calculated by Eqs. (5.15)–(5.18). The results for the normalized hoop stress resultant $aN_s/\hat{p}$ are shown in Fig. 3, from which we see that the solutions for holes are approximated by $k = 10^{-6}$. The correctness of our results in the limiting cases is therefore verified and the trend from soft inclusions to holes is also reasonable. Note that in numerical calculation, the infinite series are truncated into finite terms which are determined by truncating error defined by $\text{Tr} = [(N_s)_{k+1} - (N_s)_k]/(N_s)_k$. In the present example, if $\text{Tr} = 10^{-6}$, $\omega = 0°$, $k = 2$, the term needed is 14. When the applied load $\hat{p}$ is located on $(\hat{x}_1/a, \hat{x}_2/a) = (0, 1.1)$, the term needed is 27. To

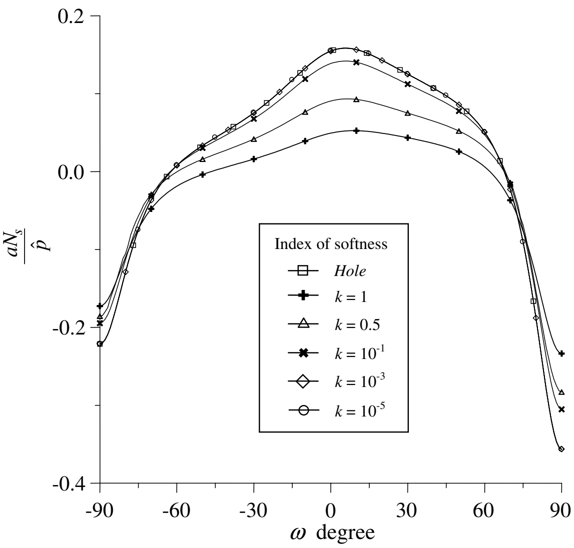

Fig. 3. Hoop stress along the elliptical inclusion boundary.

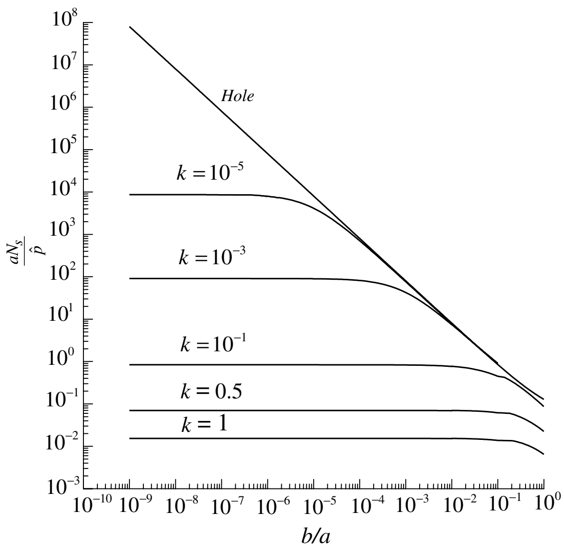

Fig. 4. Hoop stress at $\omega = 0°$ for the soft inclusion.

see the effect of elliptic shape and the singular behavior near the crack tip, a series of numerical data for the hoop stress resultant at $\omega = 0°$ are shown in Fig. 4. A nearly constant value of the hoop stress resultant for $b \to 0$ is achieved when the inclusion is not a hole, which means that no singular behavior occurs for the general elastic inclusions. For elliptical holes, singular behavior occurs when $b \to 0$ which is expected for cracks.

8. Conclusions

The analytical closed form solutions for composite laminates with elliptical elastic inclusions subjected to in-plane/out-of-plane concentrated forces and moments are obtained in this paper through the use of Stroh-like formalism and the method of analytical continuation. These solutions can be classified into six categories by three different loading types and two different loading locations. They are: (case 1) inplane concentrated forces $\hat{f}_1, \hat{f}_2$ and out-of-plane concentrated moments $\hat{m}_1, \hat{m}_2$, (case 2) out-of-plane concentrated force $\hat{f}_3$, and (case 3) in-plane torsion $\hat{m}_3$, which may locate outside the inclusion or inside the inclusion. Among these solutions, the one corresponding to the loading type of case 1 applied outside the inclusions is the most familiar solution since it has exactly the same mathematical form as that of the corresponding two-dimensional problems. Moreover, no special difficulty has been encountered for this condition than that of hole/crack problems. For the loads applied inside the inclusions, special treatment given in (4.4) and (4.5) is necessary to avoid discontinuity raised by the requirement of one-to-one transformation. To solve the problems under the loading types of cases 2 and 3, the key step that lead us to the solutions is splitting the unperturbed solution into three parts like those shown in (4.6) where the first one is holomorphic outside the unit circle, the second one is holomorphic inside the unit circle, and the third one is singular in both region. With the solutions provided in this paper, analytical solutions for any arbitrary loading can be obtained through superposition. Moreover, these solutions can serve as the fundamental solutions for boundary element formulation and as the kernel functions for the integral equations considering the interaction between inclusions and cracks. Moreover, they will also be useful for the study of homogenized elastic constitutive properties of elastic solids with micro-inclusions, voids and cracks.

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Appendix A.

A.1. Case 1:

With $\zeta_\alpha = \zeta_\alpha^* = \sigma = e^{i\omega}$ along the interface, substitution of (6.2a)$_3$ and (6.3a)$_1$ into (6.5a) will lead to

$$\begin{aligned}&\mathbf{A}_1\mathbf{d} - \overline{\mathbf{A}}_1\overline{\mathbf{d}} = (\mathbf{A}_2\mathbf{A}_2^T + \overline{\mathbf{A}}_2\overline{\mathbf{A}}_2^T)\hat{\mathbf{p}}/2\pi i,\\ &\mathbf{B}_1\mathbf{d} - \overline{\mathbf{B}}_1\overline{\mathbf{d}} = (\mathbf{B}_2\mathbf{A}_2^T + \overline{\mathbf{B}}_2\overline{\mathbf{A}}_2^T)\hat{\mathbf{p}}/2\pi i.\end{aligned} \tag{A.1}$$

, Since $i(2\mathbf{AB}^T - \mathbf{I}) = \mathbf{S}$ and $2i\mathbf{AA}^T = \mathbf{H}$ are real matrices (Ting, 1996), (A.1) can be reduced and re-organized into

$$\begin{bmatrix}\mathbf{A}_1 & \overline{\mathbf{A}}_1\\ \mathbf{B}_1 & \overline{\mathbf{B}}_1\end{bmatrix}\begin{Bmatrix}\mathbf{d}\\ -\overline{\mathbf{d}}\end{Bmatrix} = \frac{1}{2\pi i}\begin{Bmatrix}\mathbf{0}\\ \hat{\mathbf{p}}\end{Bmatrix}. \tag{A.2}$$

From the orthogonality relations of the material eigenvector matrices, we know that (Ting, 1996)

$$\begin{bmatrix}\mathbf{A}_1 & \overline{\mathbf{A}}_1\\ \mathbf{B}_1 & \overline{\mathbf{B}}_1\end{bmatrix},\begin{bmatrix}\mathbf{B}_1^T & \mathbf{A}_1^T\\ \overline{\mathbf{B}}_1^T & \overline{\mathbf{A}}_1^T\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \quad \text{or} \quad \begin{bmatrix}\mathbf{B}_1^T & \mathbf{A}_1^T\\ \overline{\mathbf{B}}_1^T & \overline{\mathbf{A}}_1^T\end{bmatrix}\begin{bmatrix}\mathbf{A}_1 & \overline{\mathbf{A}}_1\\ \mathbf{B}_1 & \overline{\mathbf{B}}_1\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}. \tag{A.3}$$

Use of the identities given in (A.3), the unknown coefficient vector $\mathbf{d}$ can easily be obtained from (A.2) through the inversion process as

$$\mathbf{d} = \frac{1}{2\pi i}\mathbf{A}_1^T\hat{\mathbf{p}}. \tag{A.4}$$

A.2. Case 2:

Like case 1, with $\zeta_\alpha = \zeta_\alpha^* = \sigma = e^{i\omega}$ along the interface, substitution of (4.1a), (6.2b)$_3$ and (6.3b)$_1$ into (6.5a) will lead to

$$\begin{aligned}&\mathbf{A}_1\{\cos\psi(\mathbf{d}_1 + \mathbf{d}_{-1}) + i\sin\psi(\mathbf{d}_1 - \mathbf{d}_{-1}) + \mathbf{d}_0\}\\ &- \overline{\mathbf{A}}_1\{\cos\psi(\overline{\mathbf{d}}_1 + \overline{\mathbf{d}}_{-1}) - i\sin\psi(\overline{\mathbf{d}}_1 - \overline{\mathbf{d}}_{-1}) + \overline{\mathbf{d}}_0\}\\ &= \frac{\hat{f}_3}{2\pi i}\{\cos\psi(\mathbf{E}_{2R} + \mathbf{F}_{2R}) - \sin\psi(\mathbf{E}_{2I} - \mathbf{F}_{2I}) - \mathbf{J}_{2R}\}\mathbf{i}_3\end{aligned} \tag{A.5a}$$

$$\begin{aligned}&\mathbf{B}_1\{\cos\psi(\mathbf{d}_1 + \mathbf{d}_{-1}) + i\sin\psi(\mathbf{d}_1 - \mathbf{d}_{-1}) + \mathbf{d}_0\}\\ &- \overline{\mathbf{B}}_1\{\cos\psi(\overline{\mathbf{d}}_1 + \overline{\mathbf{d}}_{-1}) - i\sin\psi(\overline{\mathbf{d}}_1 - \overline{\mathbf{d}}_{-1}) + \overline{\mathbf{d}}_0\}\\ &= \frac{\hat{f}_3}{2\pi i}\{\cos\psi(\mathbf{E}_{1R}^T + \mathbf{F}_{1R}^T) - \sin\psi(\mathbf{E}_{1I}^T - \mathbf{F}_{1I}^T) - \mathbf{J}_{1R}^T\}\mathbf{i}_3\end{aligned} \tag{A.5b}$$

where $\mathbf{E}_{2R}$,$\mathbf{F}_{2R}$,$\mathbf{E}_{2I}$,$\mathbf{F}_{2I}$,$\mathbf{J}_{2R}$,$\mathbf{E}_{1R}^T$, $\mathbf{F}_{1R}^T$, $\mathbf{E}_{1I}^T$, $\mathbf{F}_{1I}^T$, $\mathbf{J}_{1R}^T$ are real matrices defined in (6.8). Comparison of like terms on both sides of (A.5a) and (A.5b) will then give us

$$\begin{aligned}&\begin{bmatrix}\mathbf{A}_1 & \overline{\mathbf{A}}_1\\ \mathbf{B}_1 & \overline{\mathbf{B}}_1\end{bmatrix}\begin{Bmatrix}\mathbf{d}_1 + \mathbf{d}_{-1}\\ -(\overline{\mathbf{d}}_1 + \overline{\mathbf{d}}_{-1})\end{Bmatrix} = \frac{\hat{f}_3}{2\pi i}\begin{Bmatrix}(\mathbf{E}_{2R} + \mathbf{F}_{2R})\mathbf{i}_3\\ (\mathbf{E}_{1R}^T + \mathbf{F}_{1R}^T)\mathbf{i}_3\end{Bmatrix},\\ &\begin{bmatrix}\mathbf{A}_1 & \overline{\mathbf{A}}_1\\ \mathbf{B}_1 & \overline{\mathbf{B}}_1\end{bmatrix}\begin{Bmatrix}\mathbf{d}_1 - \mathbf{d}_{-1}\\ \overline{\mathbf{d}}_1 - \overline{\mathbf{d}}_{-1}\end{Bmatrix} = \frac{i\hat{f}_3}{2\pi i}\begin{Bmatrix}(\mathbf{E}_{2I} - \mathbf{F}_{2I})\mathbf{i}_3\\ (\mathbf{E}_{1I}^T - \mathbf{F}_{1I}^T)\mathbf{i}_3\end{Bmatrix},\\ &\begin{bmatrix}\mathbf{A}_1 & \overline{\mathbf{A}}_1\\ \mathbf{B}_1 & \overline{\mathbf{B}}_1\end{bmatrix}\begin{Bmatrix}\mathbf{d}_0\\ -\overline{\mathbf{d}}_0\end{Bmatrix} = \frac{-\hat{f}_3}{2\pi i}\begin{Bmatrix}\mathbf{J}_{2R}\mathbf{i}_3\\ \mathbf{J}_{1R}^T\mathbf{i}_3\end{Bmatrix}.\end{aligned} \tag{A.6}$$

Again, through the use of the identities (A.3), Eq. (A.6) can provide us the solutions of $\mathbf{d}_1 + \mathbf{d}_{-1}, \mathbf{d}_1 - \mathbf{d}_{-1}$ and $\mathbf{d}_0$, which will then lead to the solutions shown in (6.6b), (6.7) and (6.8).

Similarly, with $\zeta_\alpha = \zeta_\alpha^* = \sigma = e^{i\omega}$ along the interface, substitution of $(6.2b)_2$ and $(6.3b)_{2,3}$ into (6.5b) and comparison of like terms will give us the results of $\mathbf{k}_{-1}$ and $\mathbf{k}_1$ shown in (6.6b).

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